Solving Mixed Integer Programs in Practice

Linear Programming

- A linear program (LP) is an optimization problem of the form

$$
\begin{aligned}
\underset{x}{\text{minimize}} \quad & \sum_{j=1}^{n} c_j x_j \\
\text{subject to} \quad & \sum_{j=1}^{n} A_{ij} x_j = b_i, \quad i = 1, \ldots, m, \\
& \ell_j \le x_j \le u_j, \quad j = 1, \ldots, n,
\end{aligned}
$$

- Why do we care about this problem?
  - Some applications (e.g., blending in the oil industry)
  - Work horse for mixed integer programming solvers

Mixed Integer Programming

- A mixed-integer program (MIP) is an optimization problem of the form

$$
\begin{aligned}
\underset{x}{\text{minimize}} \quad & \sum_{j=1}^{n} c_j x_j \\
\text{subject to} \quad & \sum_{j=1}^{n} A_{ij} x_j = b_i, \quad i = 1, \ldots, m, \\
& \ell_j \le x_j \le u_j, \quad j = 1, \ldots, n, \\
& \boxed{\text{some or all } x_j \text{ integer}}
\end{aligned}
$$

- Why do we care about this problem?

Applications of Mixed Integer Programming

- Accounting
- Advertising
- Agriculture
- Airlines
- ATM provisioning
- Compilers
- Defense
- Electrical power
- Energy
- Finance
- Food service
- Forestry
- Gas distribution
- Government
- Internet applications
- Logistics/supply chain
- Medical
- Mining
- National research labs
- Online dating
- Portfolio management
- Railways
- Recycling
- Revenue management
- Semiconductor
- Shipping
- Social networking
- Sports betting
- Sports scheduling
- Statistics
- Steel Manufacturing
- Telecommunications
- Transportation
- Utilities
- Workforce scheduling
- ...

MIP Application Types

- Static MIP
  - Formulate problem
  - Solve it with a black-box MIP algorithm
  - Read solution
  - Potentially adjust problem and iterate
  - most frequent use of MIP in practical applications
- Branch-and-cut
  - Problem has too many constraints to formulate in static fashion
    - e.g., classical TSP model: exponentially many sub-tour elimination constraints
  - Construct partial problem
  - Add violated constraints on demand
- Branch-and-price
  - Problem has too many variables to formulate in static fashion
    - e.g., many public transport and airline problems are solved via B&P
  - Start with subset of variables
  - Pricing: add variables that may improve solution on the fly
  - Usually needs problem specific branching rule that is compatible with pricing
  - Heuristic variant: column generation
    - Only apply pricing for the root LP, then solve static MIP with resulting set of variables

MIP Building Blocks

- Presole
  - Tighten formulation and reduce problem size
- Solve continuous relaxations
  - Ignoring integrality
  - Gives a bound on the optimal integral objective
- Cutting planes
  - Cut off relaxation solutions
- Branching variable selection
  - Crucial for limiting search tree size
- Primal heuristics
  - Find integer feasible solutions

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LP Presolve

- Goal
  - Reduce the problem size
    - Speedup linear algebra during the solution process
- Example

$$x + y + z \leq 5 \qquad (1)$$
$$u - x - z = 0 \qquad (2)$$
$$\cdots\cdots\cdots$$
$$0 \leq x, y, z \leq 1 \qquad (3)$$
$$u \text{ is free} \qquad (4)$$

- Reductions
  - Redundant constraint
    - (3) $\Rightarrow$ $x + y + z \leq 3$, so (1) is redundant
  - Substitution
    - (2) and (4) $\Rightarrow$ u can be substituted with $x + z$

MIP Presolve

- Goals:
  - Reduce problem size
    - Speed-up linear algebra during the solution process
  - Strengthen LP relaxation
  - Identify problem sub-structures
    - Cliques, implied bounds, networks, disconnected components, ...
- Similar to LP presolve, but more powerful:
  - Exploit integrality
    - Round fractional bounds and right hand sides
    - Lifting/coefficient strengthening
    - Probing
  - Does not need to preserve duality
    - We only need to be able to uncrush a primal solution
    - Neither a dual solution nor a basis needs to be uncrushed

MIP Presolve

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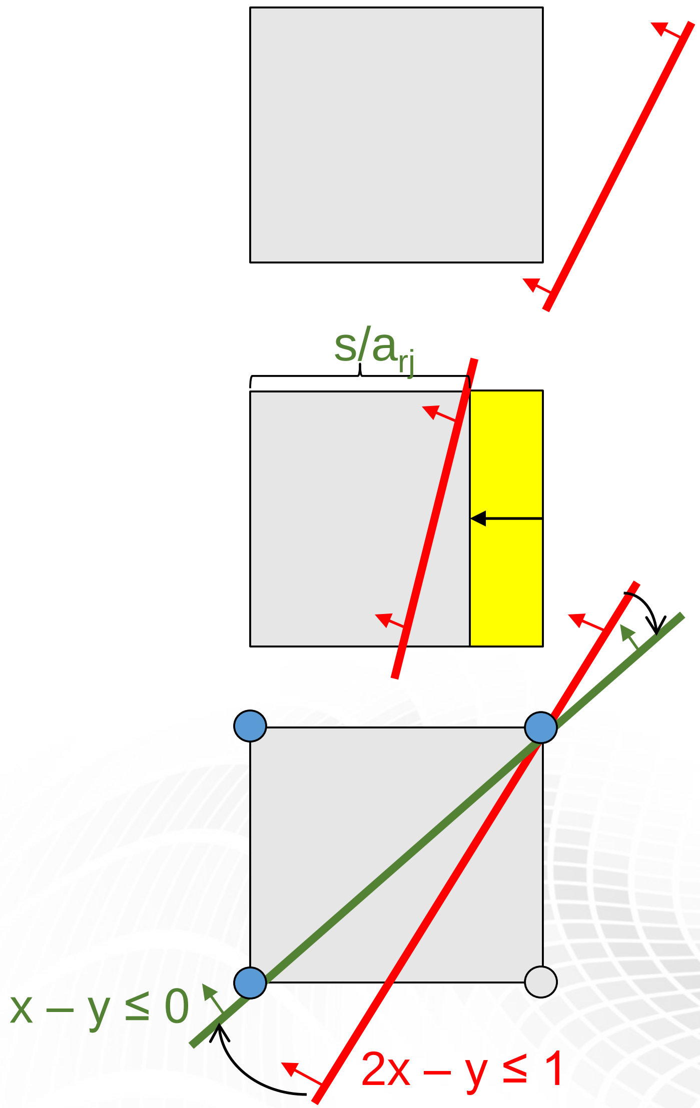
    - We
    - Neit

model	without presolve			with presolve		
	rows	cols	LP obj	rows	cols	LP obj
roll3000	2291	1166	11097.1	987	855	11120.0
neos-787933	1897	236376	3.0	41	126	30.0

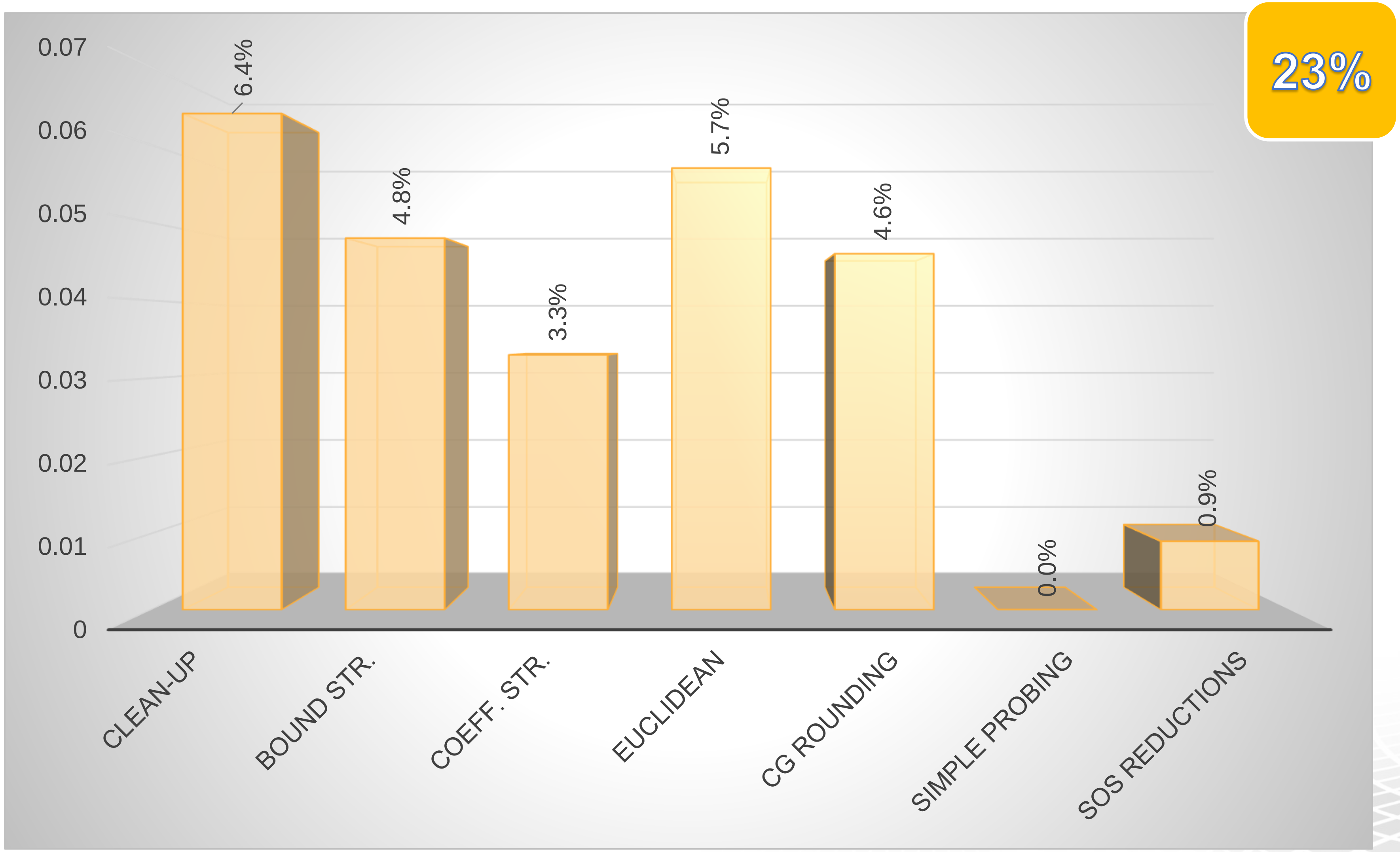
Single-Row Reductions

- Clean-up rows
  - Discard empty rows
  - Discard redundant inequalities: $\sup\{A_{r.}x\} \leq b_r$
  - Remove coefficients with tiny impact $|a_{ij} \cdot (u_j - l_j)|$
- Bound strengthening
  - $a_{rj} > 0$, $s := b_r - \inf\{A_{r.}x\} \Rightarrow x_j \leq l_j + s/a_{rj}$
  - $a_{rj} < 0$, $s := b_r - \inf\{A_{r.}x\} \Rightarrow x_j \geq u_j + s/a_{rj}$
- Coefficient strengthening for inequalities
  - $j \in I$, $a_{rj} > 0$, $t := b_r - \sup\{A_{r.}x\} + a_{rj} > 0$

    $\Rightarrow a_{rj} := a_{rj} - t$, $b_r := b_r - u_j t$



Single-Row Reductions – Performance

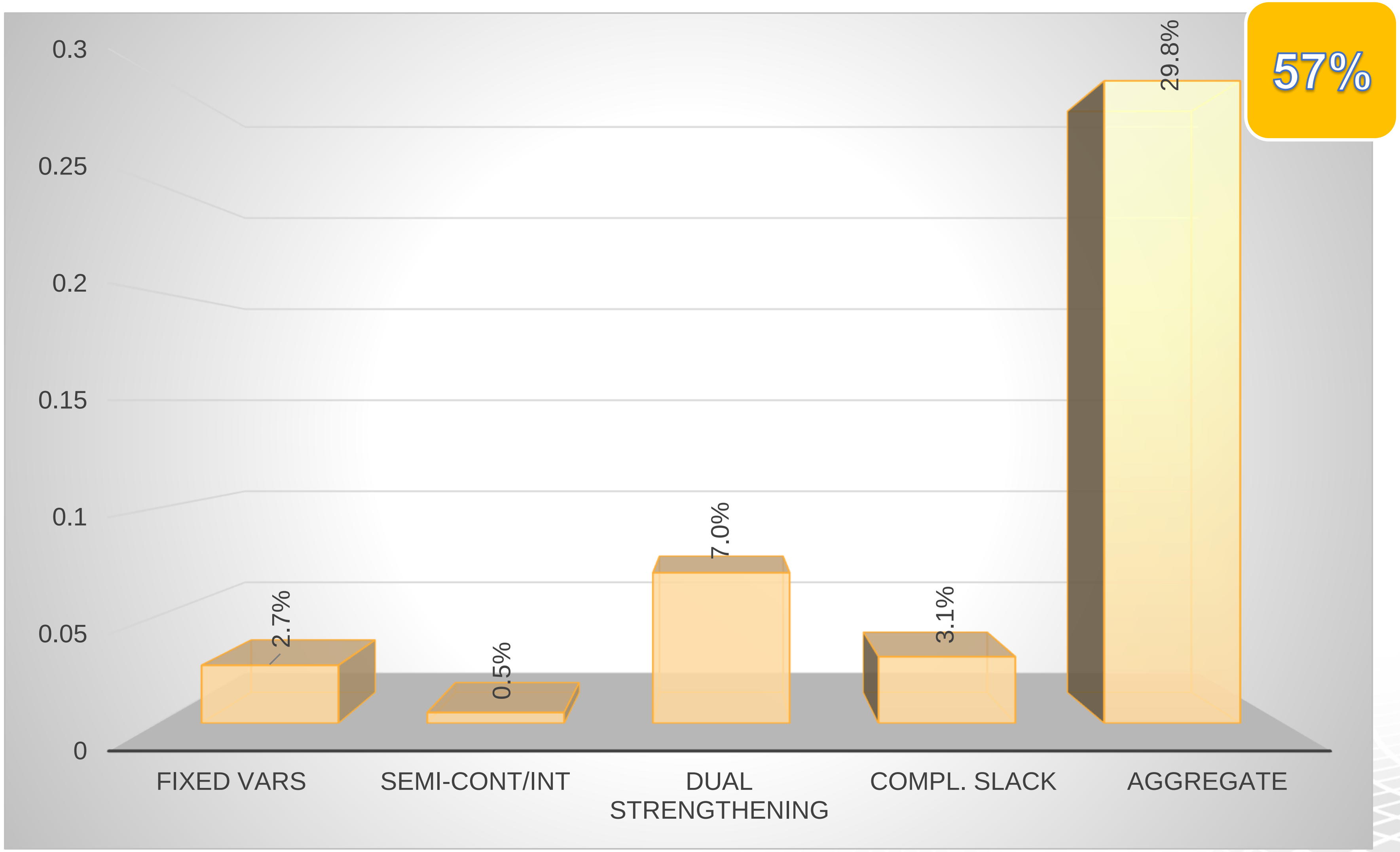


benchmark data based on Gurobi 5.6

Single-Column Reductions

- Remove fixed variables and empty columns
  - If $|u_j - l_j| \le \epsilon$, fix to some value in $[l_j, u_j]$ and move terms to rhs
  - Choice of value can be very tricky for numerical reasons
- Round bounds of integer variables
- Strengthen semi-continuous and semi-integer variables
- Dual fixing, substitution, and bound strengthening
  - Variable x_j does not appear in equations
  - $c_j \ge 0$, $A_{\cdot j} \ge 0 \quad \Rightarrow \quad x_j := l_j$
  - $\left.\begin{array}{l} c_j \ge 0,\ A_{\cdot j} \ge 0 \text{ except for } a_{ij} < 0, \\ z = 0 \rightarrow \text{row } i \text{ redundant}, \\ z = 1 \rightarrow x_j = u_j \end{array}\right\} \quad \Rightarrow \quad x_j := l_j + (u_j - l_j)\cdot z$
  - $c_j \ge 0$, all rows i with $a_{ij} < 0$ redundant for $x_j \ge t \quad \Rightarrow \quad x_j \le \max\{l_j, t\}$

Single-Column Reductions – Performance



benchmark data based on Gurobi 5.6

Multi-Row Reductions

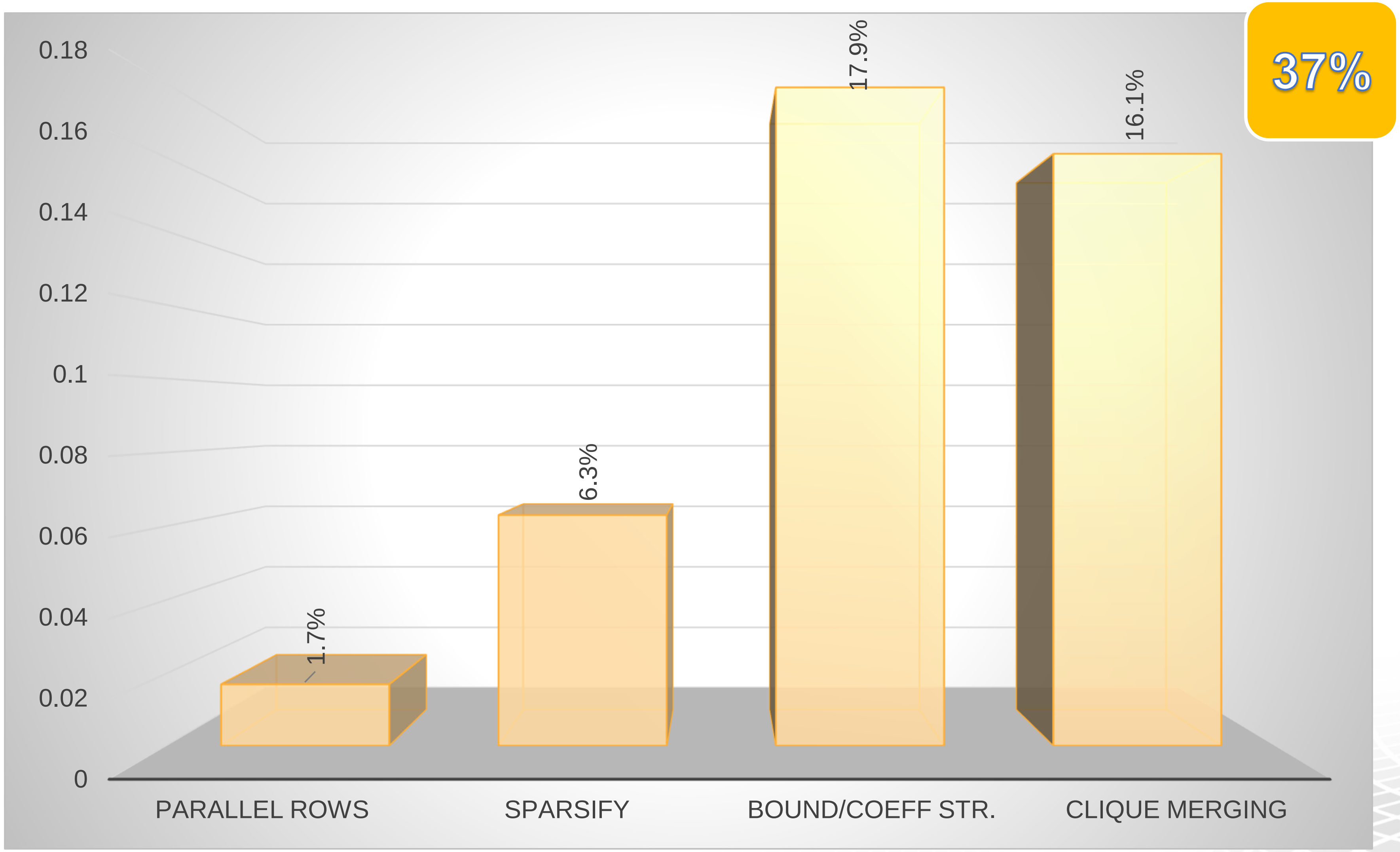
- Parallel rows
  - Search for pairs of rows such that coefficient vectors are parallel to each other
  - Discard the dominated row, or merge two inequalities into an equation
- Sparsify
  - Add equations to other rows in order to cancel non-zeros
  - Can also add inequalities with explicit slack variables
- Multi-row bound and coefficient strengthening
  - Like single-row version, but use other rows to get tighter bound on infimum and supremum $\Rightarrow$ tighter bounds, better coefficients
- Clique merging
  - Merge multiple cliques into larger single clique, e.g.:

$$\begin{array}{ccccccc} x_1 & + & x_2 & & & \leq 1 \\ x_1 & & & + & x_3 & \leq 1 \\ & & x_2 & + & x_3 & \leq 1 \end{array}$$

with binary variables x_1, x_2, x_3 can be merged into

$$x_1 + x_2 + x_3 \leq 1$$

Multi-Row Reductions

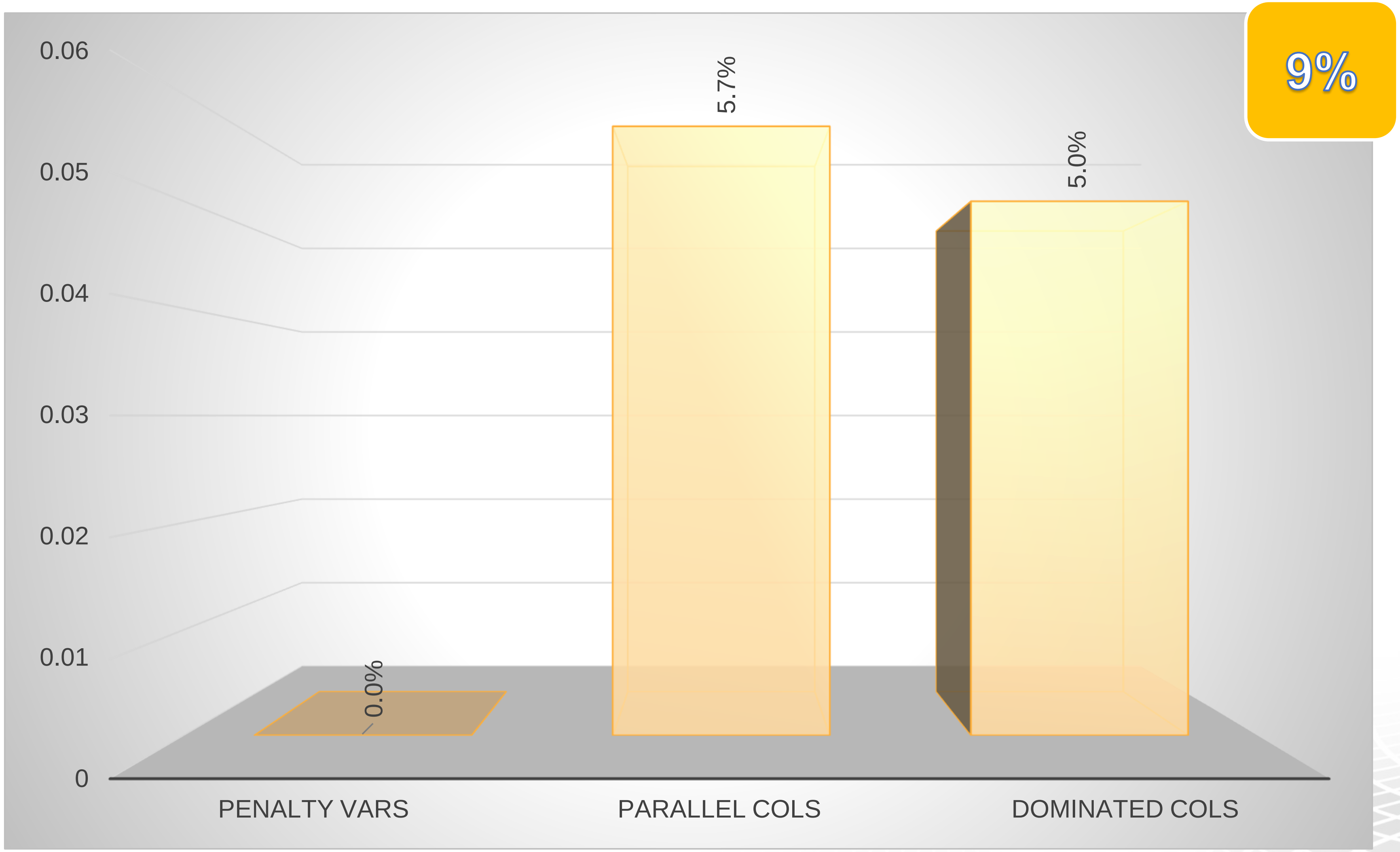


benchmark data based on Gurobi 5.6

Multi-Column Reductions

- Fix redundant penalty variables
  - Penalty variables: $\text{support}(A_{\cdot j}) = 1$
  - Multiple penalty variables in a single constraint
    - Some can be fixed if others can accomplish all that is needed
- Parallel columns (say, columns 1 and 2): $A_{\cdot 1} = sA_{\cdot 2}$
  - $u_2 = \infty$, $c_1 \geq sc_2$, $2 \notin I$ or $(|s| = 1, \{1,2\} \subseteq I)$: $x_1 := l_1$
  - $l_2 = -\infty$, $c_1 \leq sc_2$, $2 \notin I$ or $(|s| = 1, \{1,2\} \subseteq I)$: $x_1 := u_1$
  - $c_1 = sc_2$, $1,2 \notin I$ or $(|s| = 1, \{1,2\} \subseteq I)$: $x_{1'} := x_1 + sx_2$
  - Detection algorithm: two level hashing plus sorting
- Dominated columns: $A_{\cdot 1} \geq sA_{\cdot 2}$, only inequalities
  - $u_2 = \infty$, $c_1 \geq sc_2$, $2 \notin I$ or $(|s| = 1, \{1,2\} \subseteq I)$: $x_1 := l_1$
  - Detection algorithm: essentially pair-wise comparison
    - Can be very expensive: needs work limit

Multi-Column Reductions – Performance

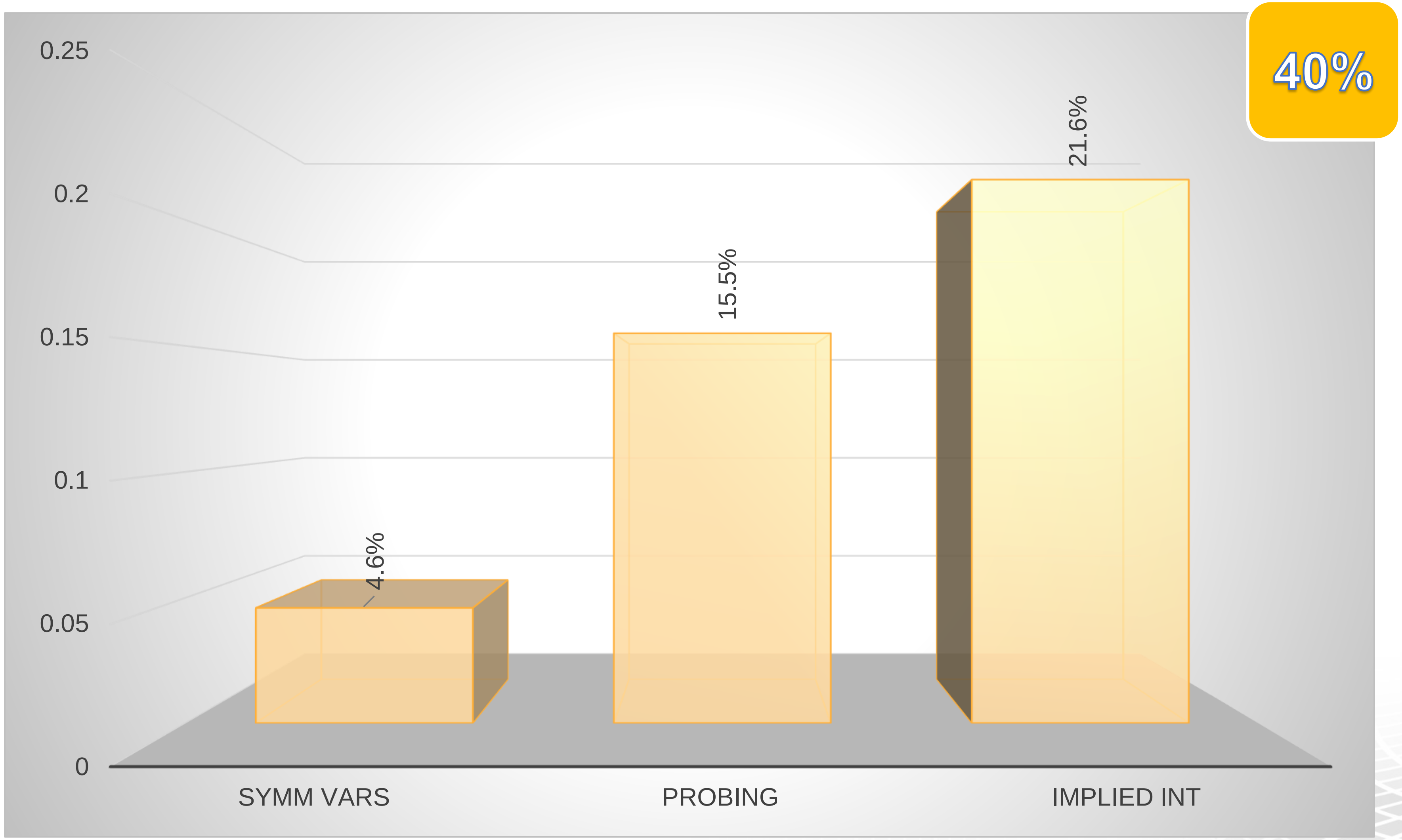


benchmark data based on Gurobi 5.6

Full Problem Reductions

- Symmetric variable substitution
  - Integer variables in same orbit can be aggregated if the involved symmetries do not overlap
  - Continuous variables in same orbit can always be aggregated
  - Issue: symmetry detection can sometimes be time consuming!
- Probing
  - Tentatively fix binary $x = 0$ and $x = 1$, propagate fixing to get domain reductions for other variables
    - $x = 0 \rightarrow y \leq u_0$, $x = 1 \rightarrow y \leq u_1 \Rightarrow y \leq \max\{u_0, u_1\}$ (bound strength.)
    - $x = 0 \rightarrow y = l_y$, $x = 1 \rightarrow y = u_y \Rightarrow y := l_y + (u_y - l_y) \cdot x$ (substitution)
    - $ay \leq b$, $x = 1 \rightarrow ay \leq d < b \Rightarrow ay + (b-d) \cdot x \leq b$ (lifting)
  - Sequence dependent
  - Can be very time consuming
    - Needs specialized data structures and algorithms
- Implied Integer Detection
  - Primal version: $ax + y = b$, x integer variables, $a \in \mathbb{Z}^n$, $b \in \mathbb{Z} \Rightarrow y$ integer
  - Dual version:
    - One of the inequalities for y will be tight, but do not know which
    - If all those inequalities lead to primal version of implied integer detection, y is implied integer

Full Problem Reductions



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MIP Building Blocks

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Primal and Dual LP

- Primal Linear Program:

$$\begin{array}{llll} \min & c^T x & & \\ s.t. & Ax & = & b \\ & x & \geq & 0 \end{array}$$

- Weighted combination of constraints (y) and bounds (z) yields

$$y^T Ax + z^T x \geq y^T b \quad (\text{with } z \geq 0)$$

- Dual Linear Program:

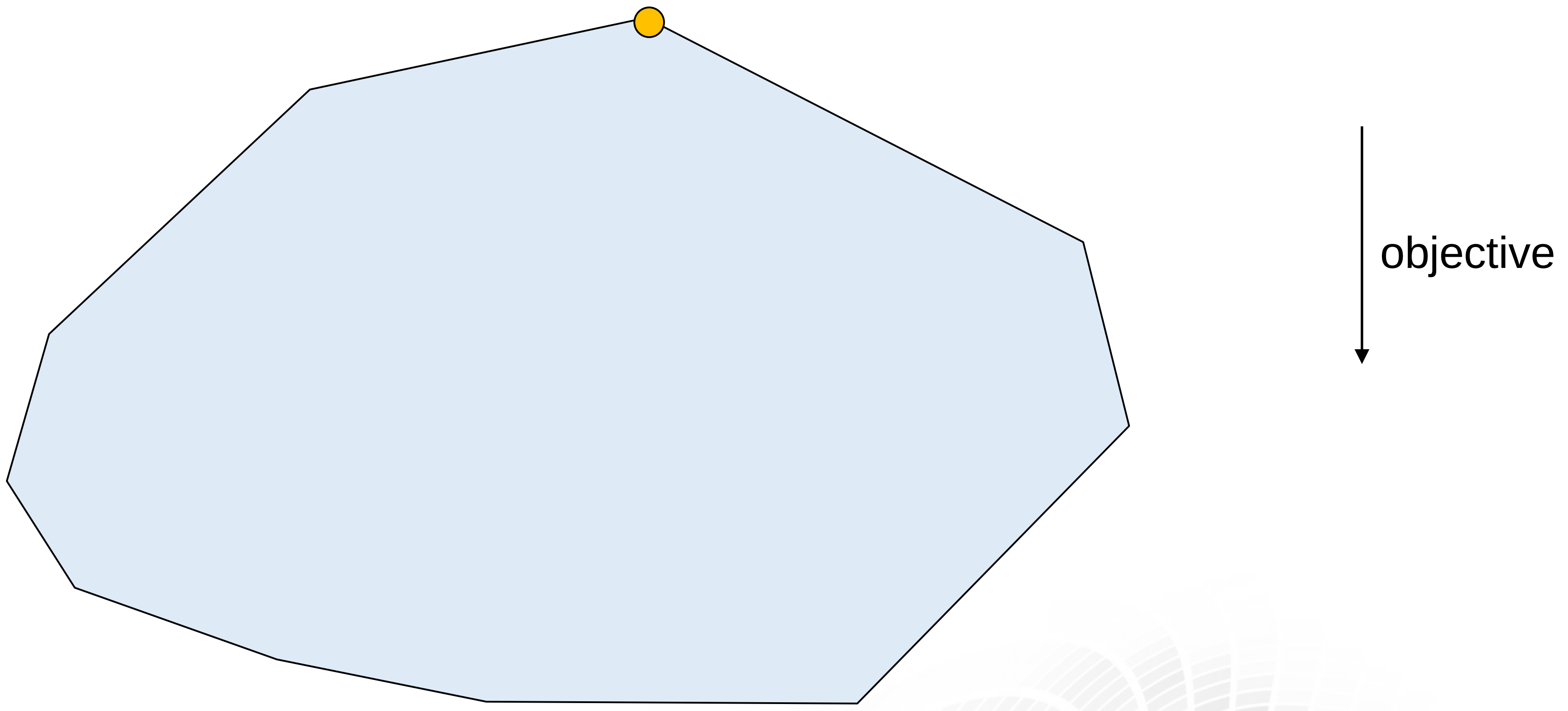
$$\begin{array}{llll} \max & y^T b & & \\ s.t. & y^T A + z^T & = & c^T \\ & z & \geq & 0 \end{array}$$

<u>Strong Duality Theorem:</u>

$$c^T x^* = y^{*T} b$$

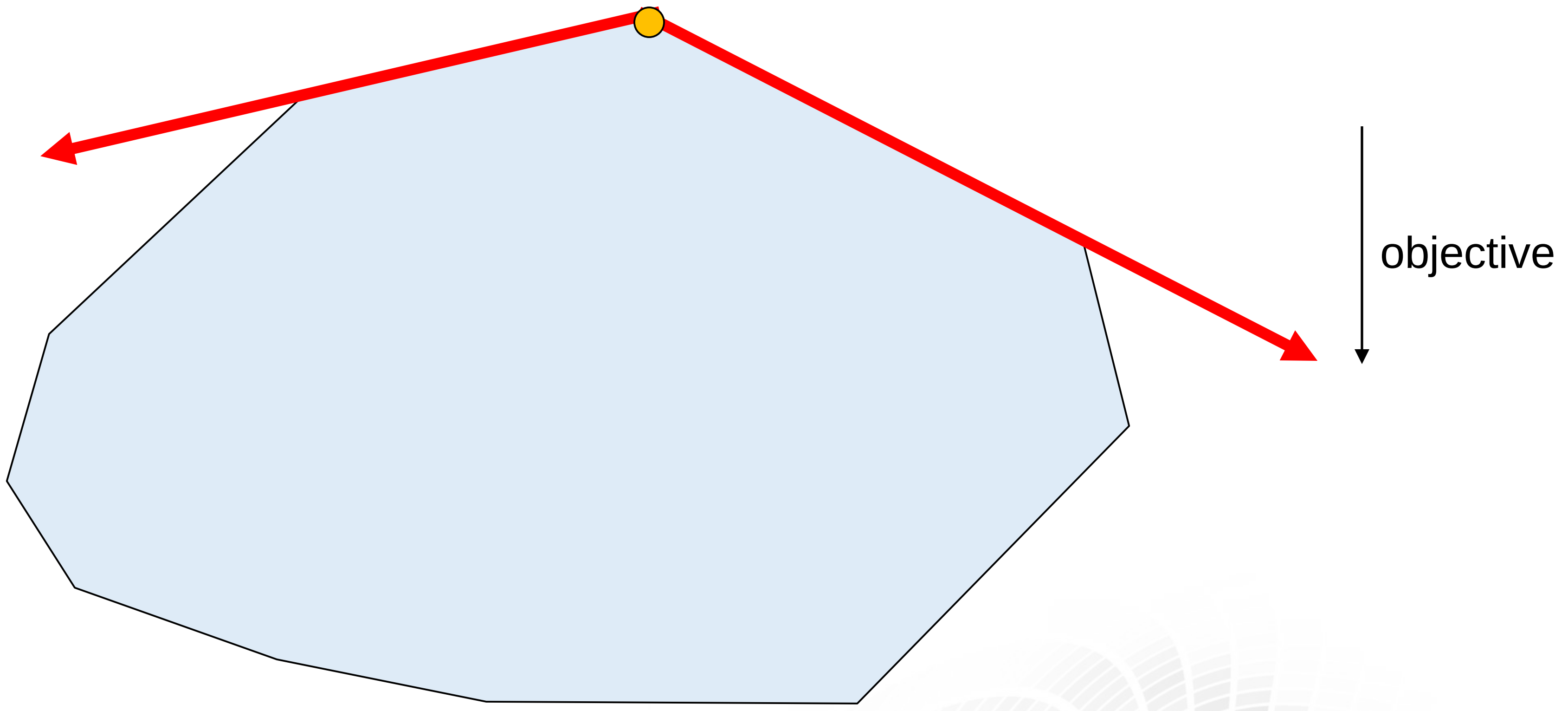
(if primal and dual are both feasible)

Simplex Algorithm



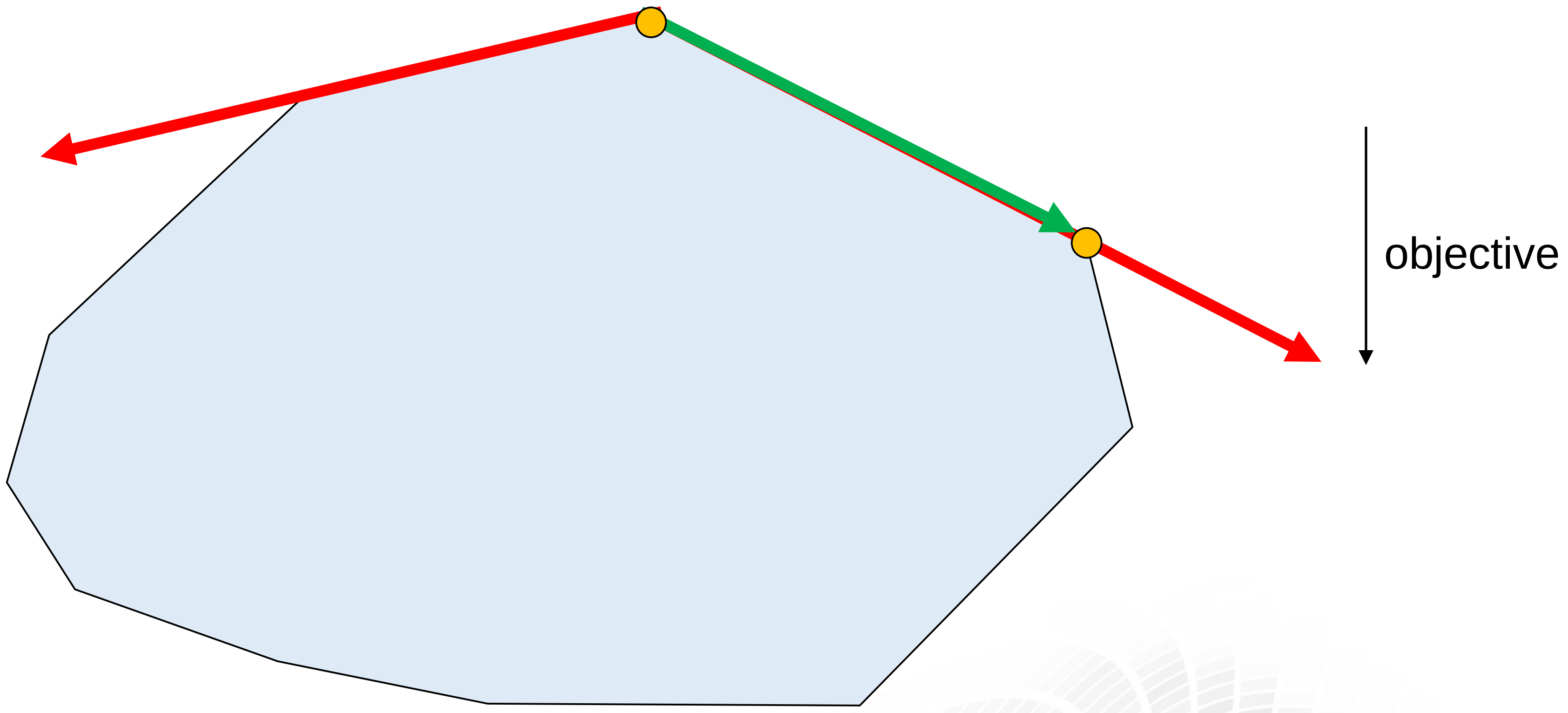
- **Phase 1**: find some feasible vertex solution

Simplex Algorithm



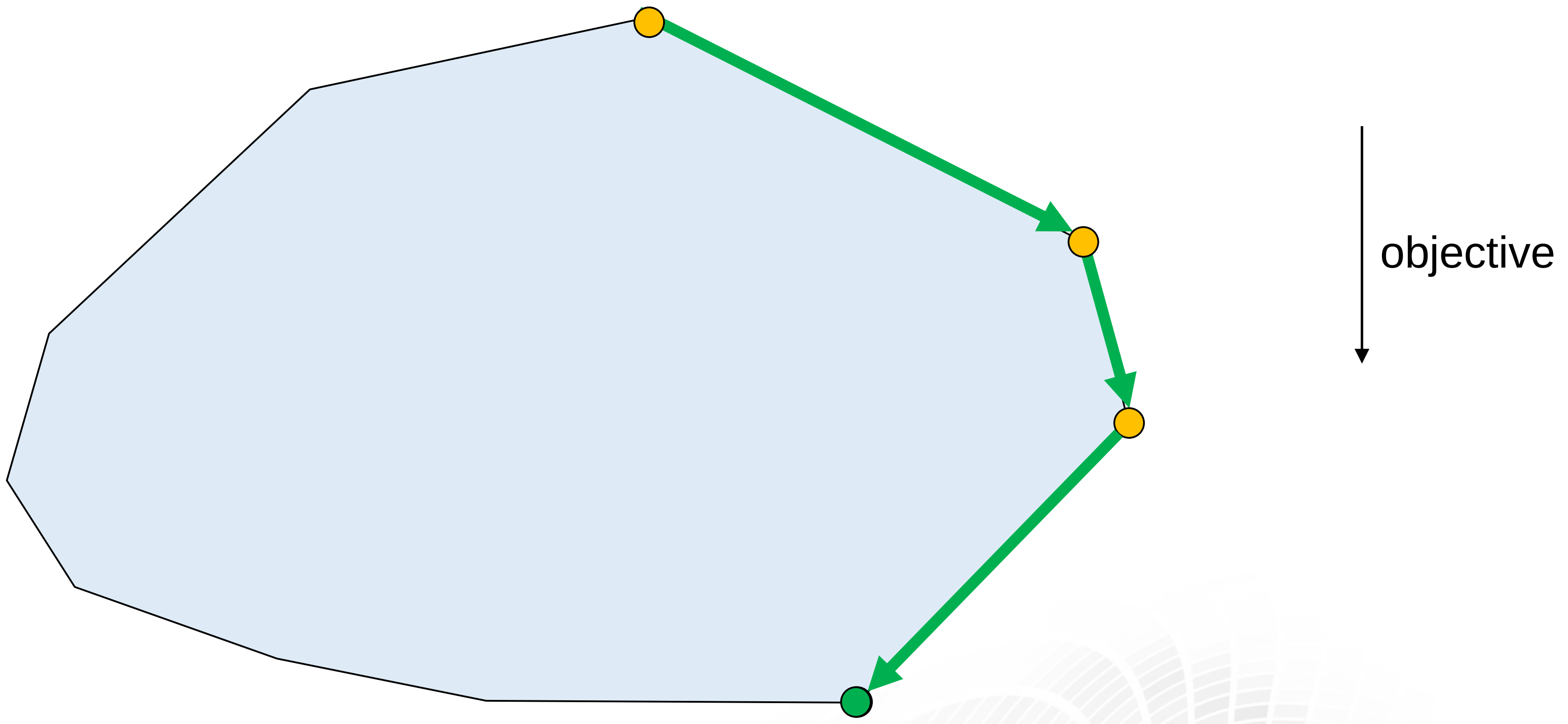
- **Pricing**: find directions in which objective improves and select one of them

Simplex Algorithm



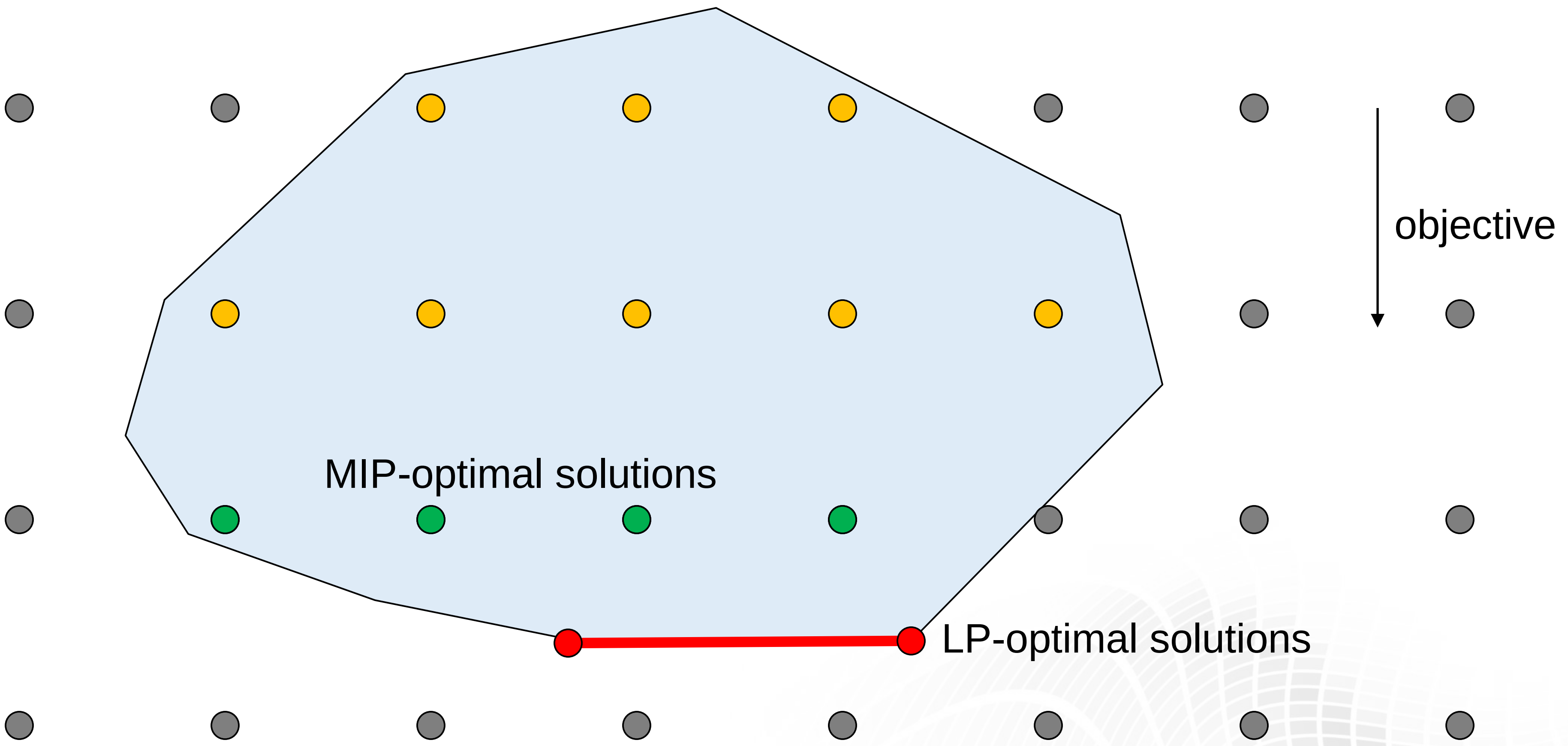
- **Ratio test**: follow outgoing ray until next vertex is reached

Simplex Algorithm



- Iterate until no more improving direction is found

MIP – LP Relaxation



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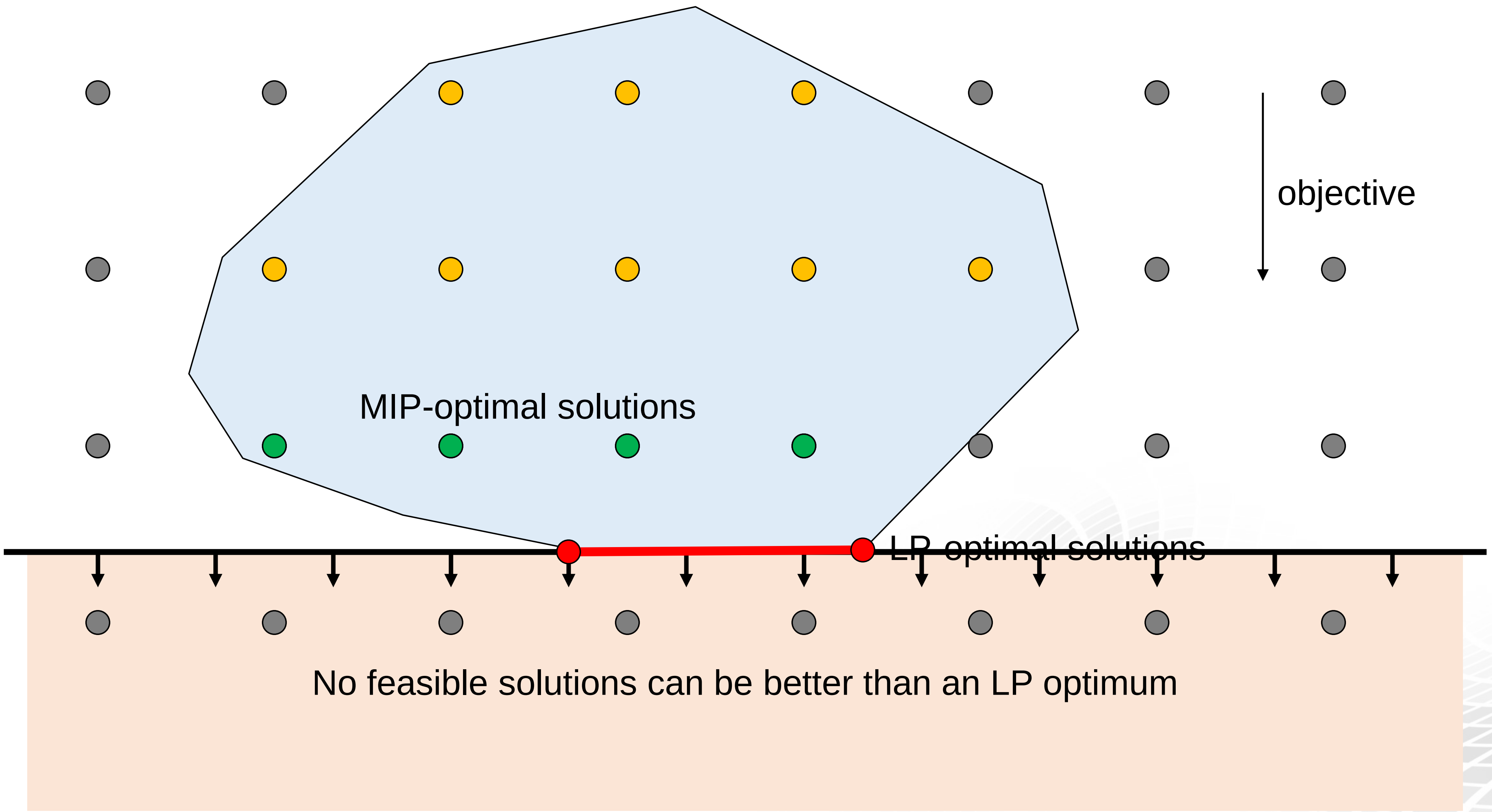

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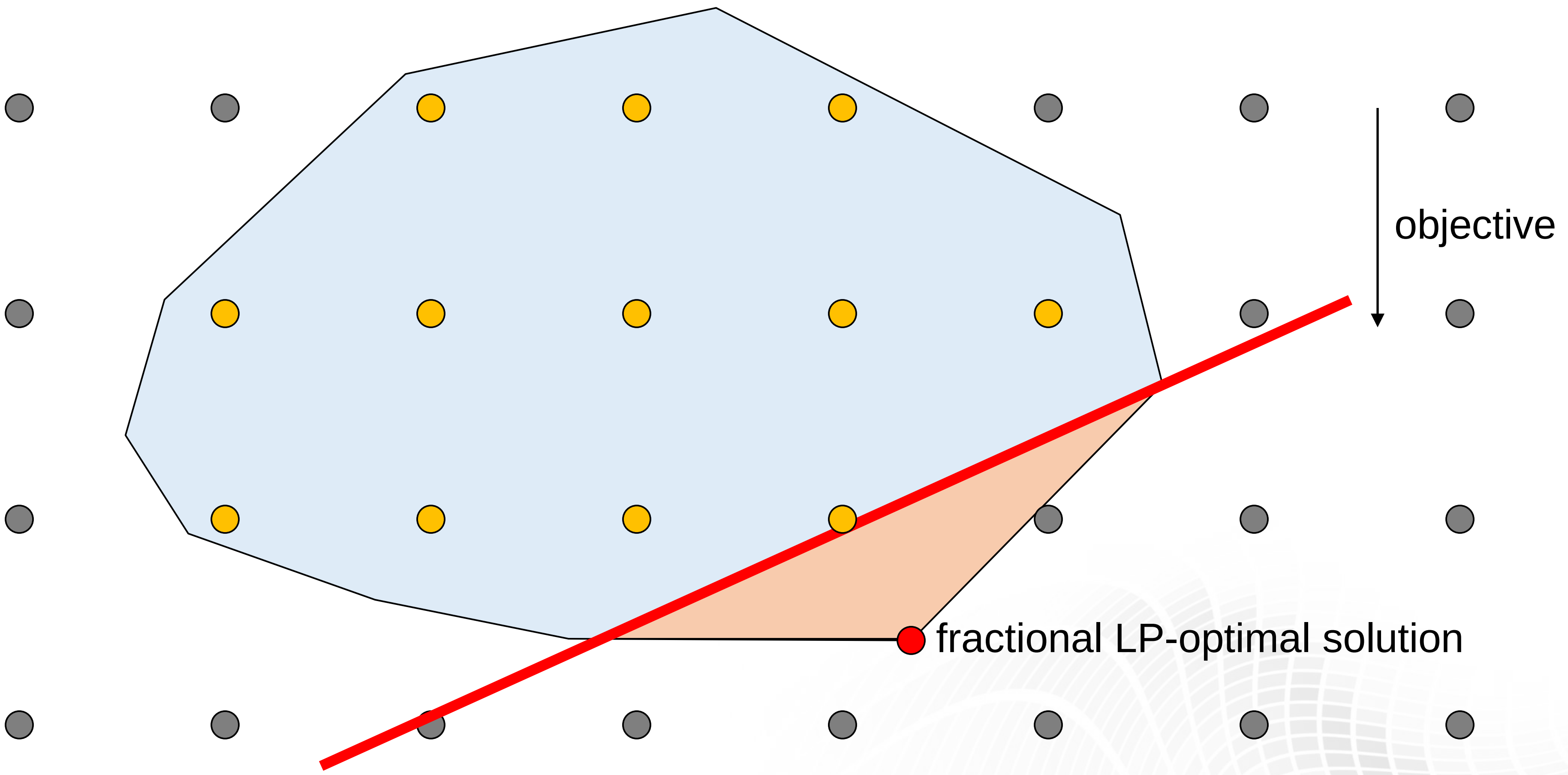
MIP – Cutting Planes



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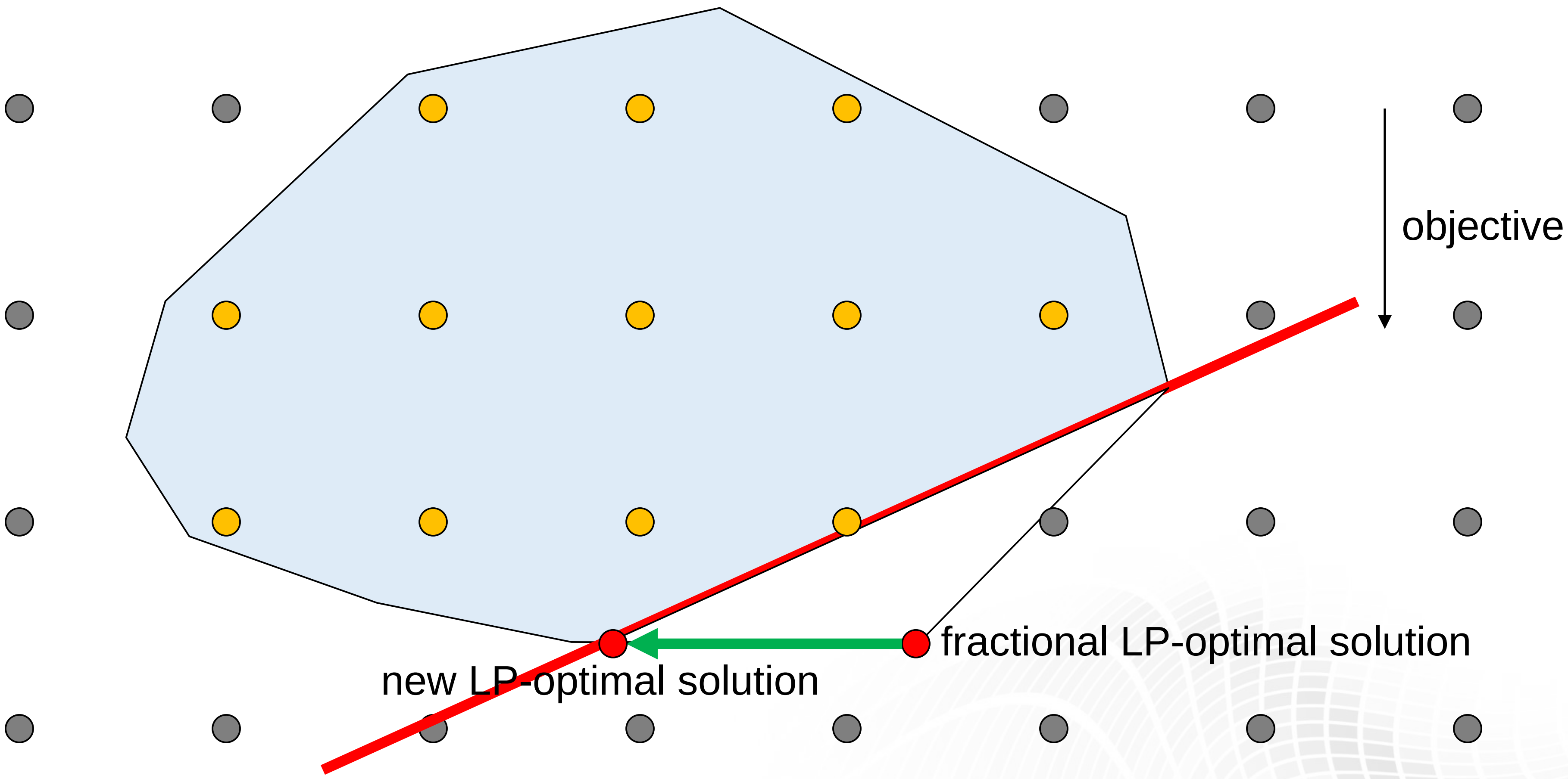

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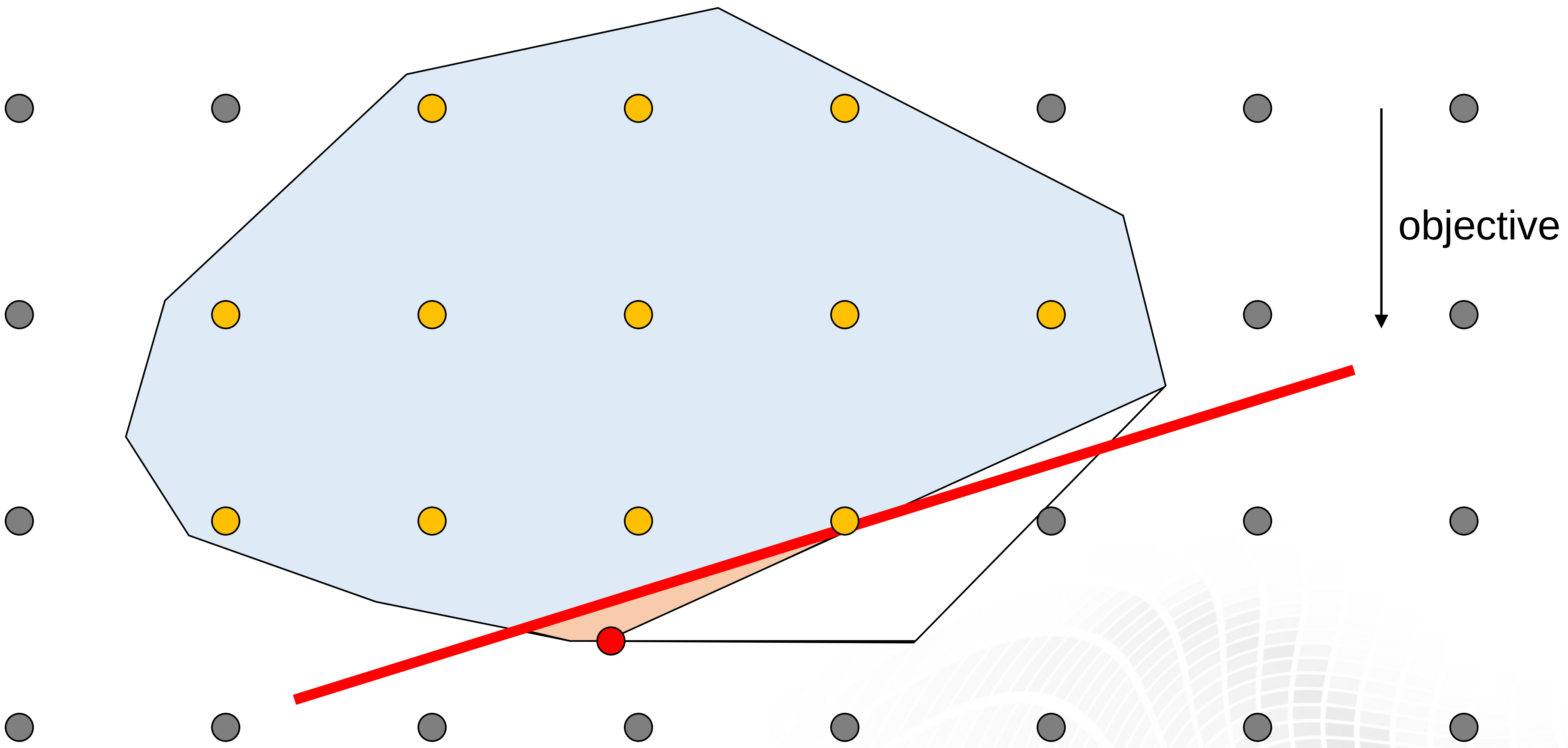

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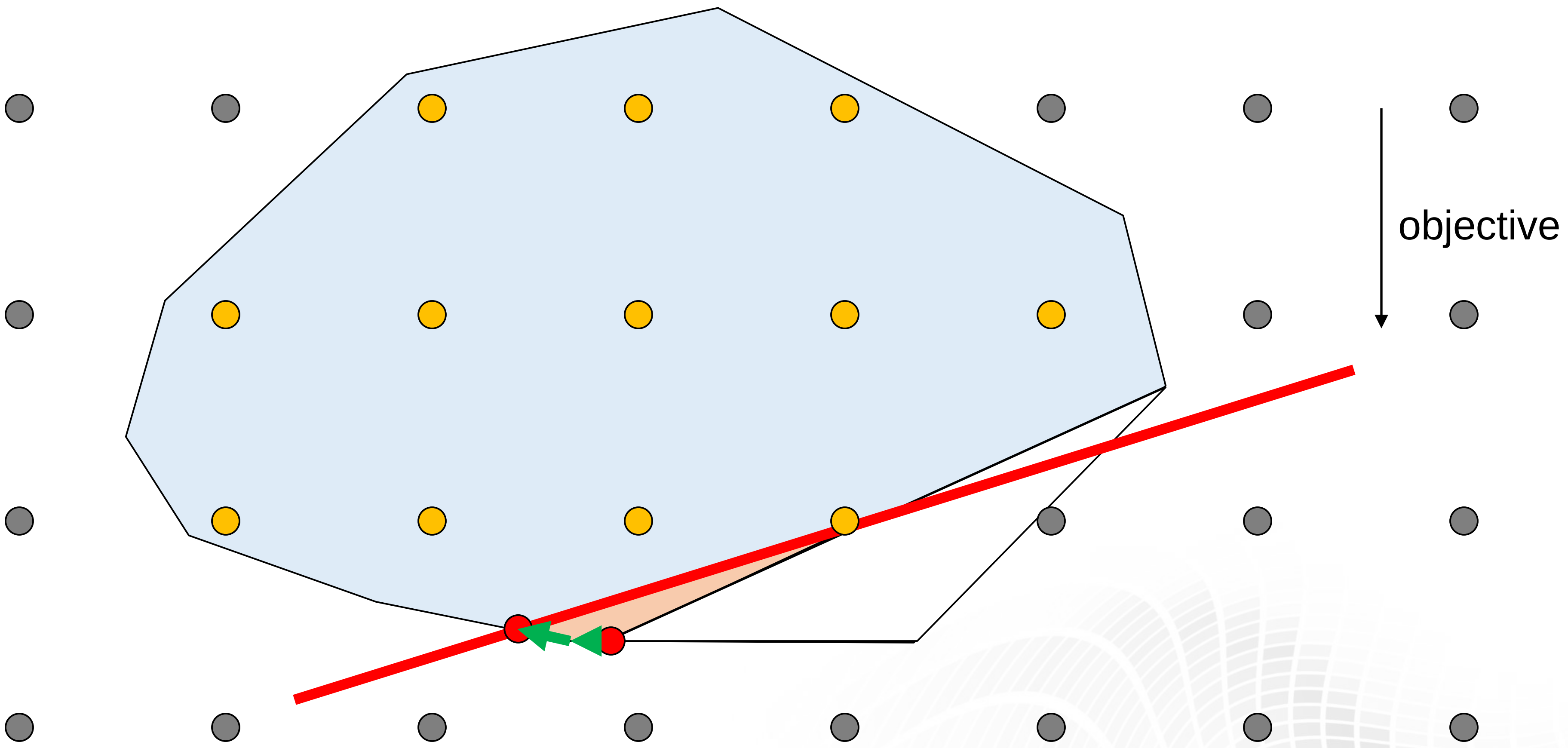

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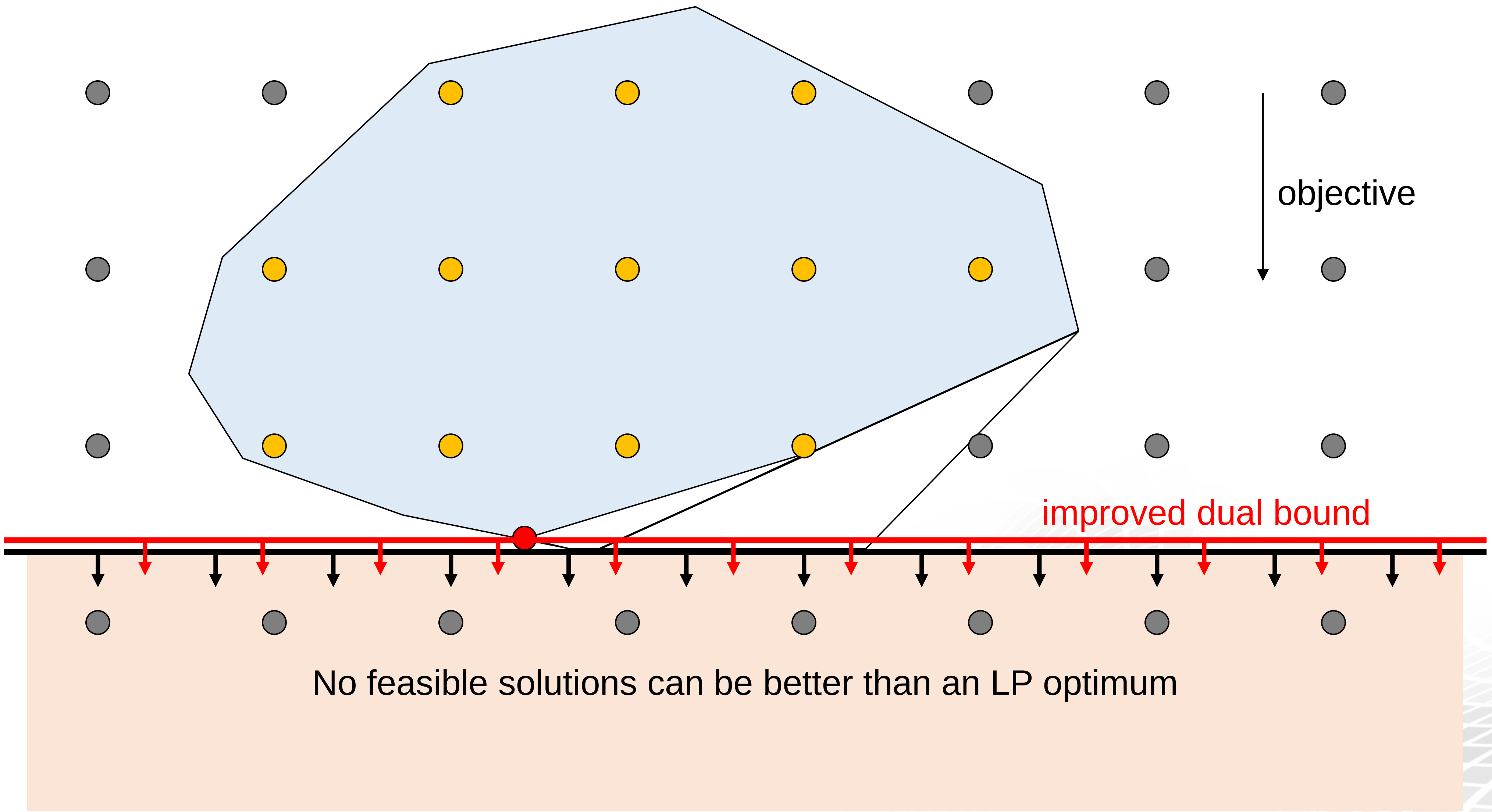


Cutting Planes – Overview

- General-purpose cutting planes
  - Gomory mixed integer cuts
  - Mixed Integer Rounding (MIR) cuts
  - Flow cover cuts
  - Lift-and-project (L&P) cuts
  - Zero-half and mod-k cuts
  - ...
- Structural cuts
  - Implied bound cuts
  - Knapsack cover cuts
  - GUB cover cuts
  - Clique cuts
  - Multi-commodity-flow (MCF) cuts
  - Flow path cuts
  - ...

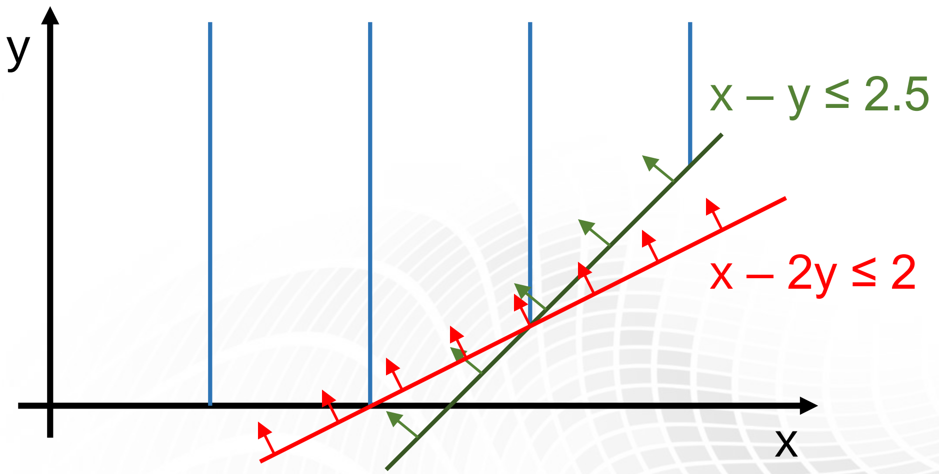
Mixed Integer Rounding Cuts

- Consider $S := \{(x,y) \in \mathbb{Z} \times \mathbb{R}_{\geq 0} \mid x - y \leq b\}$.

  Then, $$x - \frac{1}{1-f_0} y \leq \lfloor b \rfloor$$

  is valid for S with $f_0 := b - \lfloor b \rfloor$.

- Example: $x - y \leq 2.5$
- MIR cut: $x - 2y \leq 2$



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  is valid for S with $f_0 := b - \lfloor b \rfloor$.

- Consider $S := \{(x,y) \in \mathbb{Z}^p_{\geq 0}\times\mathbb{R}^q_{\geq 0} \mid ax + dy \leq b\}$.

  Then, $$\sum \left(\lfloor a_i \rfloor + \frac{max\{f_i - f_0, 0\}}{1-f_0} \right) x_i + \sum \left(\frac{min\{d_j, 0\}}{1-f_0} \right) y_j \leq \lfloor b \rfloor$$

  is valid for S with $f_i := a_i - \lfloor a_i \rfloor$, $f_0 := b - \lfloor b \rfloor$.

Mixed Integer Rounding Cuts

- General idea:
  1. Choose non-negative multipliers $\lambda \in \mathbb{R}^m_{\geq 0}$
  2. Aggregated inequality $\lambda^T Ax \leq \lambda^T b$ is valid for P because $\lambda \geq 0$
  3. Apply MIR formula to aggregated inequality to produce cutting plane
- Cut separation procedure of Marchand and Wolsey (1998, 2001):
  1. Start with one constraint of the problem (do this for each one), call this the "current aggregated inequality"
  2. Apply MIR procedure to current aggregated inequality
     (a) Complement variables if LP solution is closer to upper bound
     (b) For each a_j in constraint and each of $\delta \in \{1,2,4,8\}$ divide the constraint by $\delta|a_j|$ and apply MIR formula to resulting scaled constraint
     (c) Choose most violated cut from this set of MIR cuts
     (d) Check if complementing one more (or one less) variable yields larger violation
  3. If no violated cut was found (and did not yet reach aggregation limit):
     (a) Add another problem constraint to the current aggregated inequality such that a continuous variable with LP value not at a bound is canceled
     (b) Go to 2

Gomory Mixed Integer Cuts

- Just an alternative way to aggregate constraints
- Read them from an optimal simplex tableau:
  - Let i be a basis index with $x_i^* \notin \mathbb{Z}$
  - Choose $\lambda^T = (A_B^{-1})_{i\cdot}$
  - Resulting aggregated inequality: $x_i + (A_B^{-1})_{i\cdot} A_N\, x_N \leq (A_B^{-1})_{i\cdot} b$
- Apply MIR formula on resulting aggregated inequality
- In theory, always produces a violated cutting plane
- Practical issues:
  - Gomory Mixed Integer Cuts can be pretty dense
  - Numerics (in particular for higher rank cuts) can be very challenging
- But:
  - If done right, GMICs (together with MIRs) are currently the most important cutting planes in practice

Knapsack Cover Cuts

- A (binary) knapsack is a constraint $ax \leq b$ with
  - $a_i \geq 0$ the weight of item i, $i = 1,\ldots,n$
  - $b \geq 0$ the capacity of the knapsack
- An index set $C \subseteq \{1,\ldots,n\}$ is called a *cover*, if $\sum_{i \in C} a_i > b$
- A cover C entails a cover inequality

$$\sum_{i \in C} x_i \leq |C| - 1$$

- Interesting for cuts: minimal covers

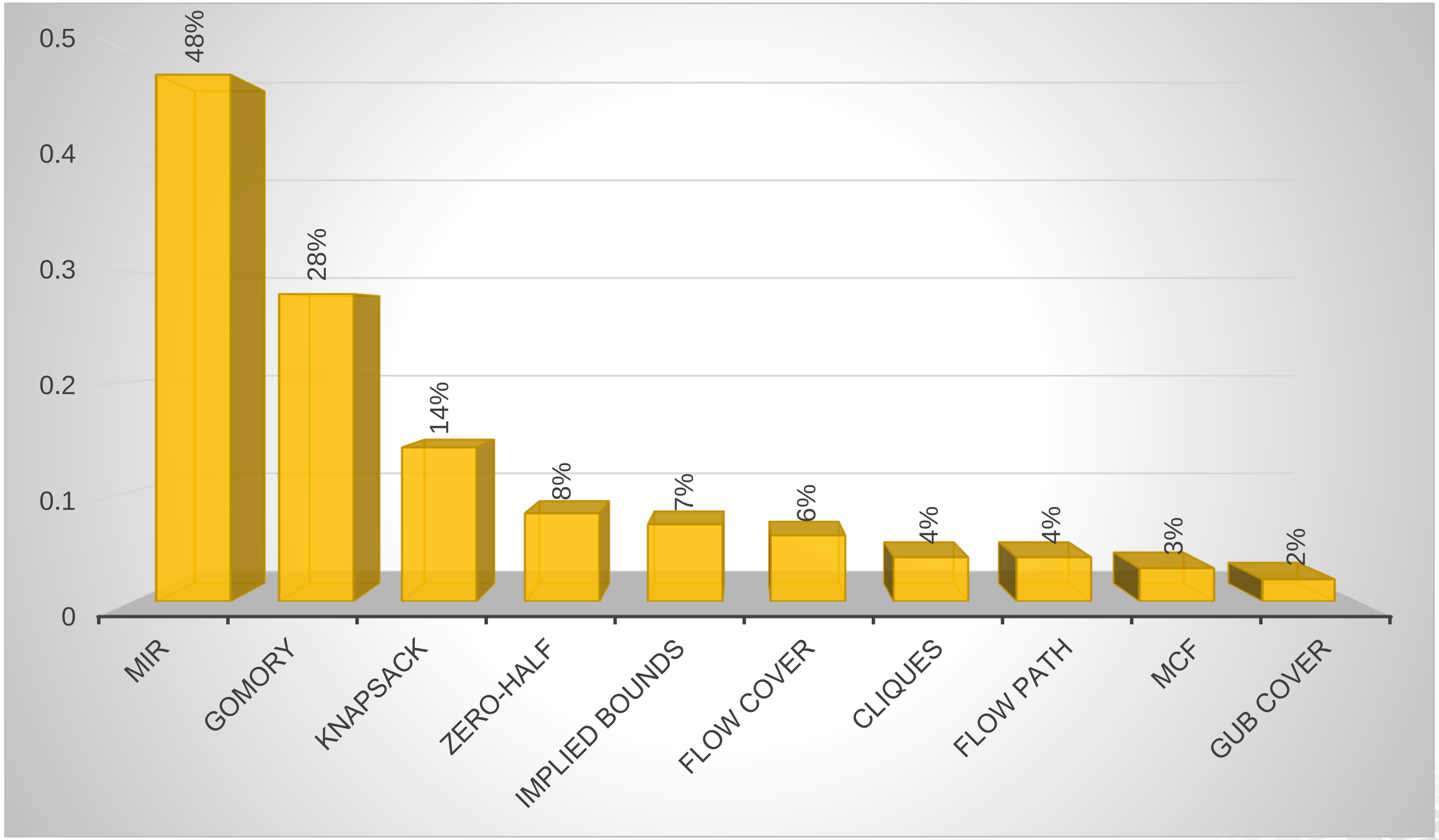
$$\sum_{i \in C} a_i > b \quad \text{and} \quad \sum_{i \in C'} a_i \leq b \quad \text{for all} \quad C' \subsetneq C$$

Knapsack Cover Cuts – Example

- Consider knapsack $3x_1 + 5x_2 + 8x_3 + 10x_4 + 17x_5 \leq 24,\ x \in \{0,1\}^5$
- A minimal cover is $C = \{1,2,3,4\}$
- Resulting cover inequality: $x_1 + x_2 + x_3 + x_4 \leq 3$

- Lifting
  - If $x_5 = 1$, then $x_1 + x_2 + x_3 + x_4 \leq 1$
  - Hence, $x_1 + x_2 + x_3 + x_4 + 2x_5 \leq 3$ is valid
  - Need to solve knapsack problem $\alpha_j := d_0 - \max\{dx \mid ax \leq b - a_j\}$ to find lifting coefficient for variable x_j
    - Use dynamic programming to solve knapsack problem

Cutting Planes – Performance



Achterberg and Wunderling: "Mixed Integer Programming: Analyzing 12 Years of Progress" (2013)
benchmark data based on CPLEX 12.5

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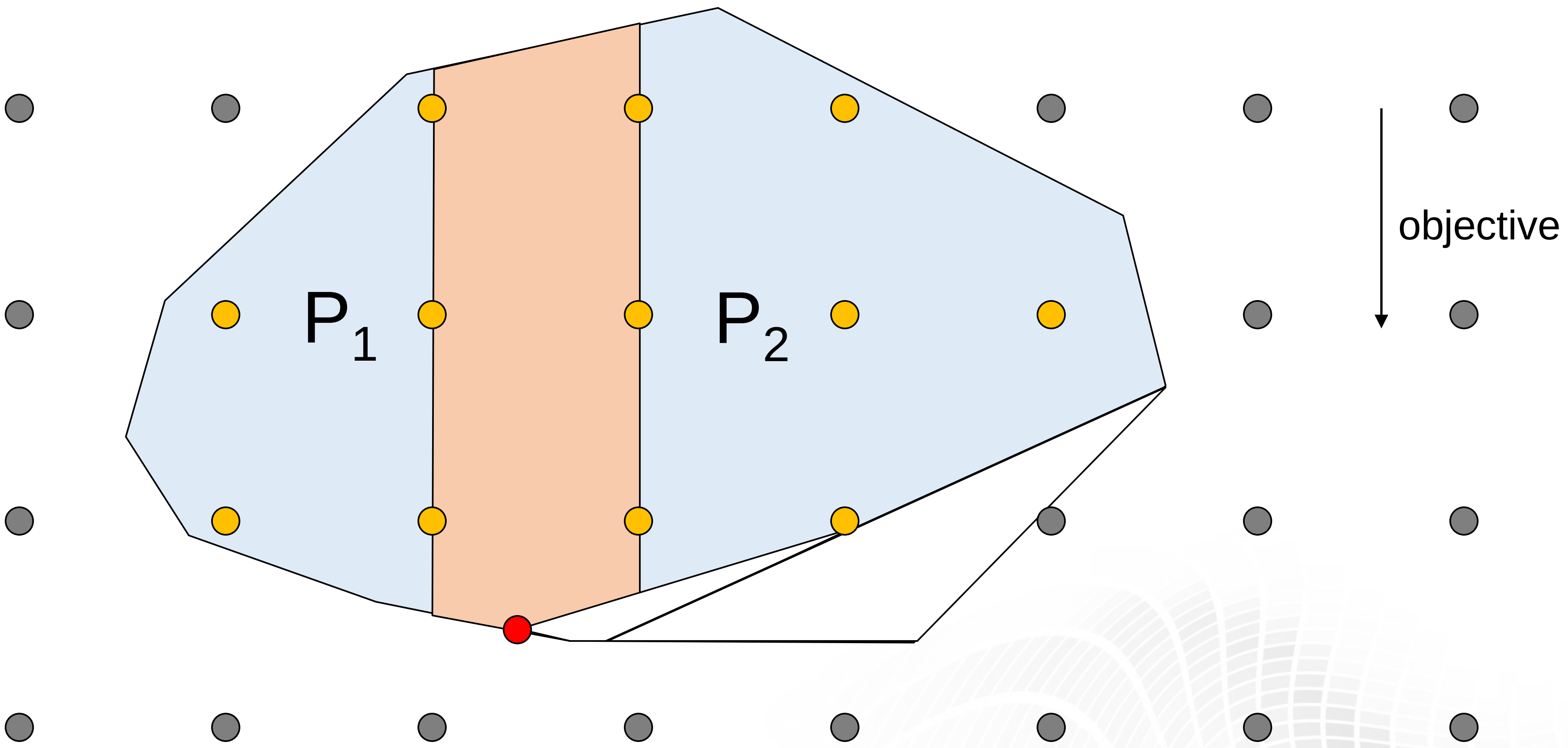
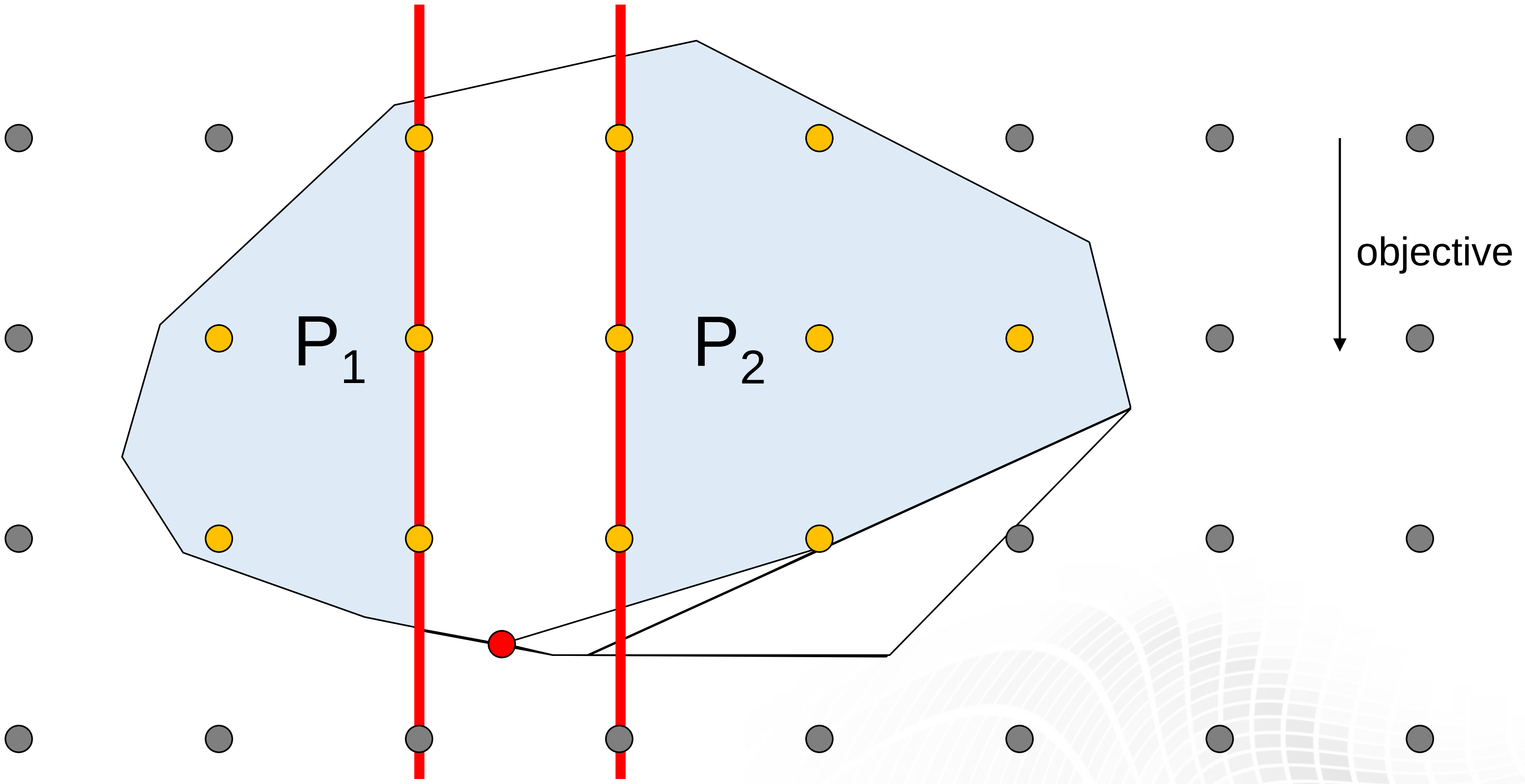
MIP – Branching



MIP – Branching



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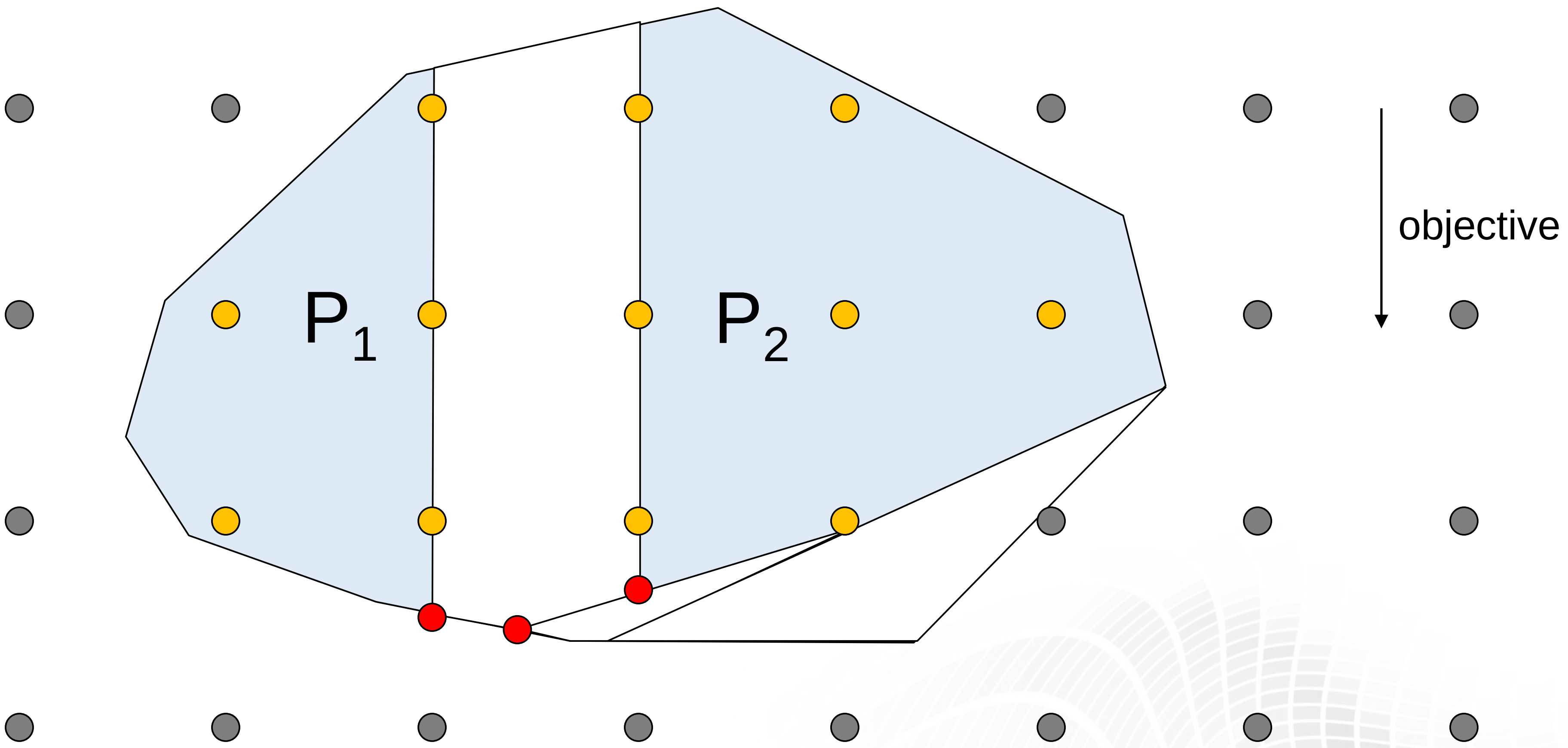
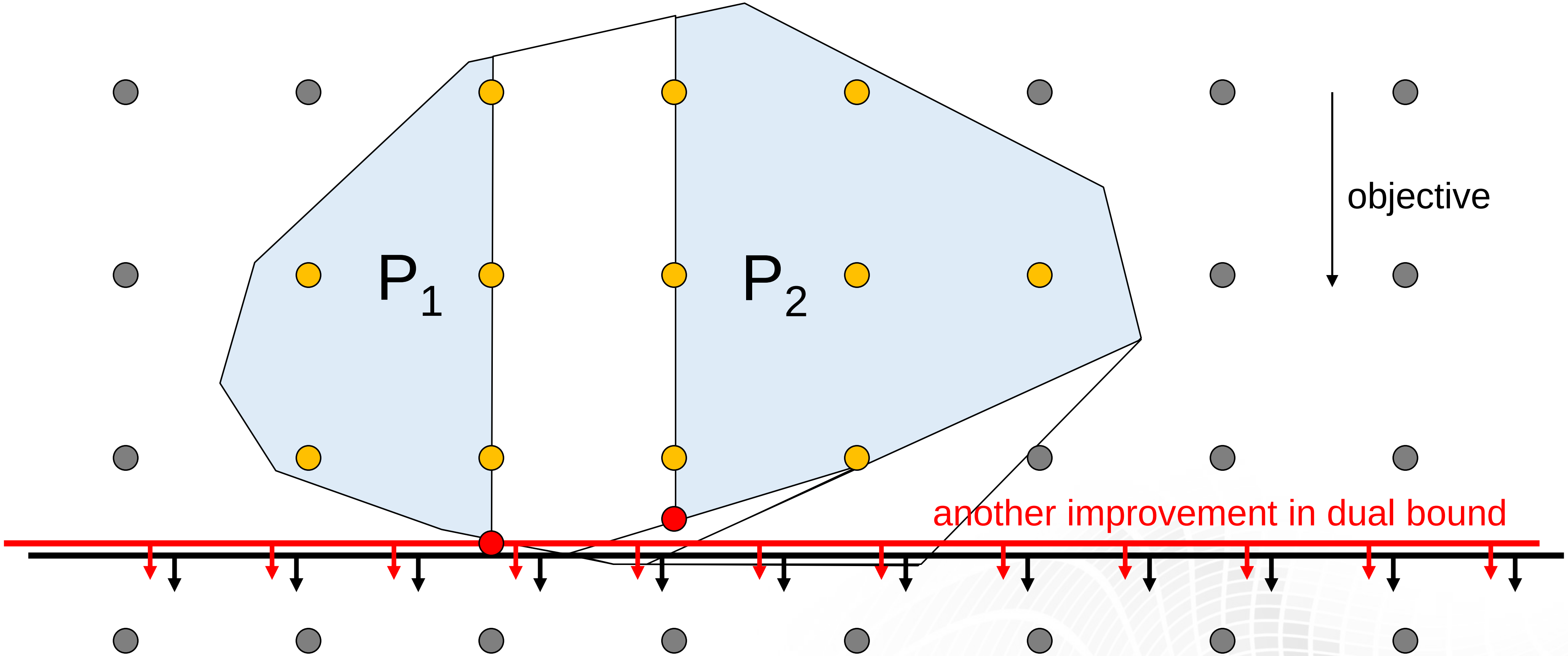
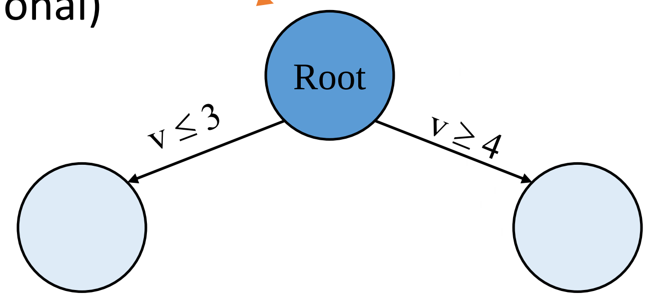


MIP – Branching

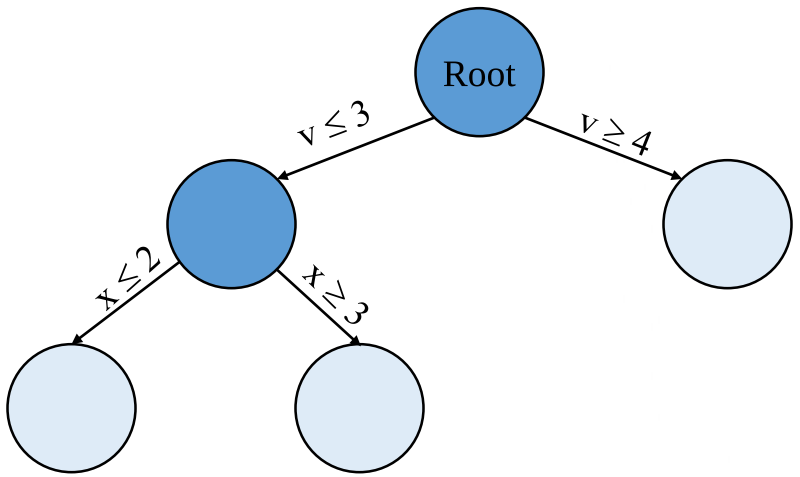


LP based Branch-and-Bound

Solve LP relaxation:
v=3.5 (fractional)



LP based Branch-and-Bound



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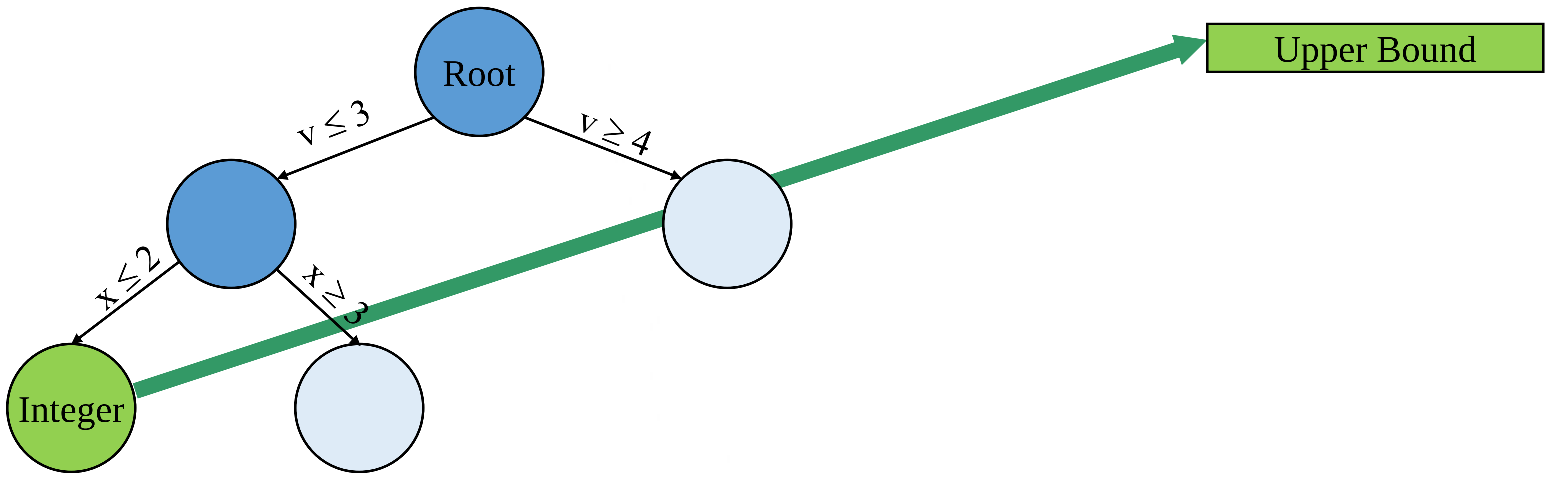
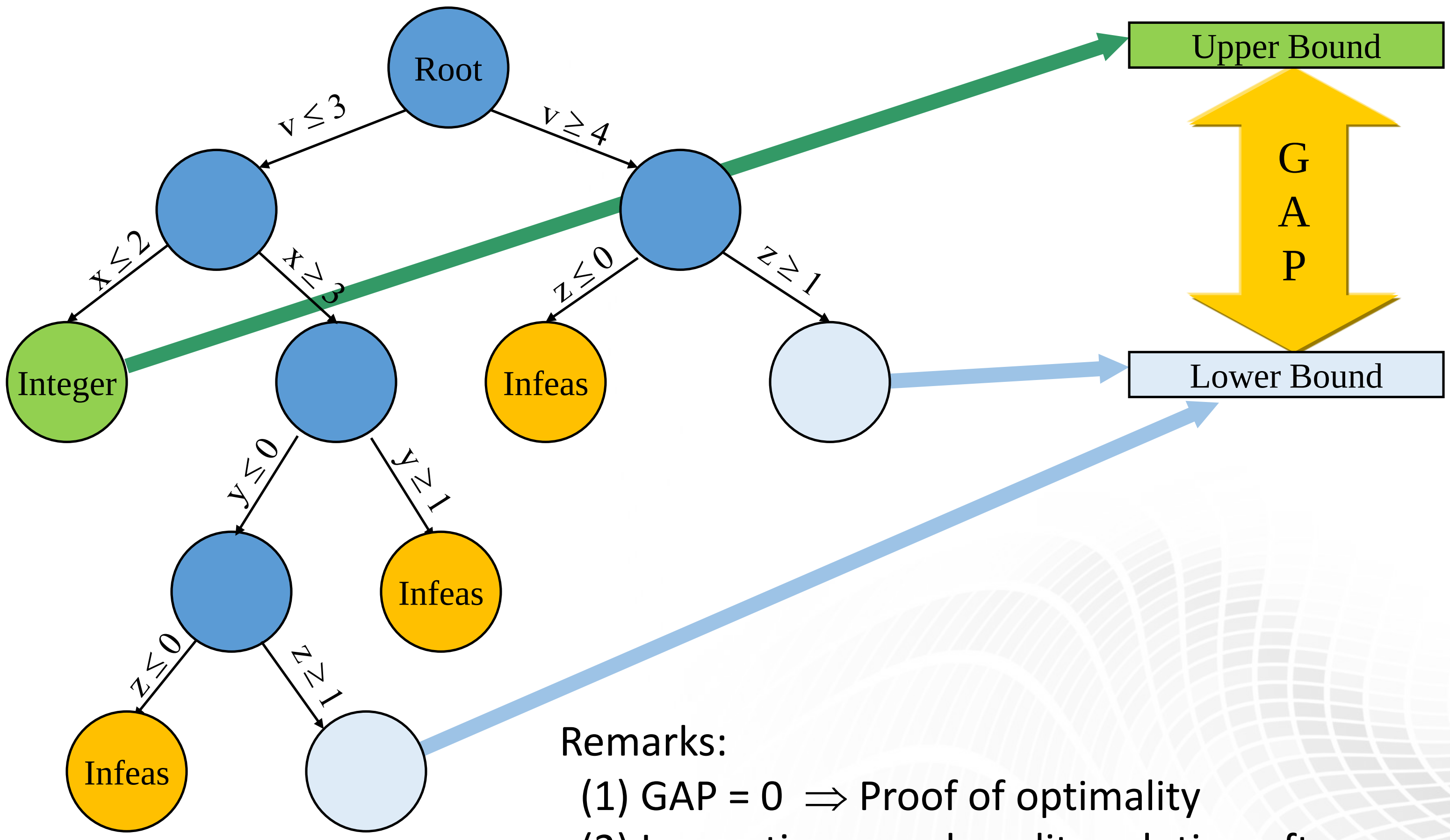


LP based Branch-and-Bound

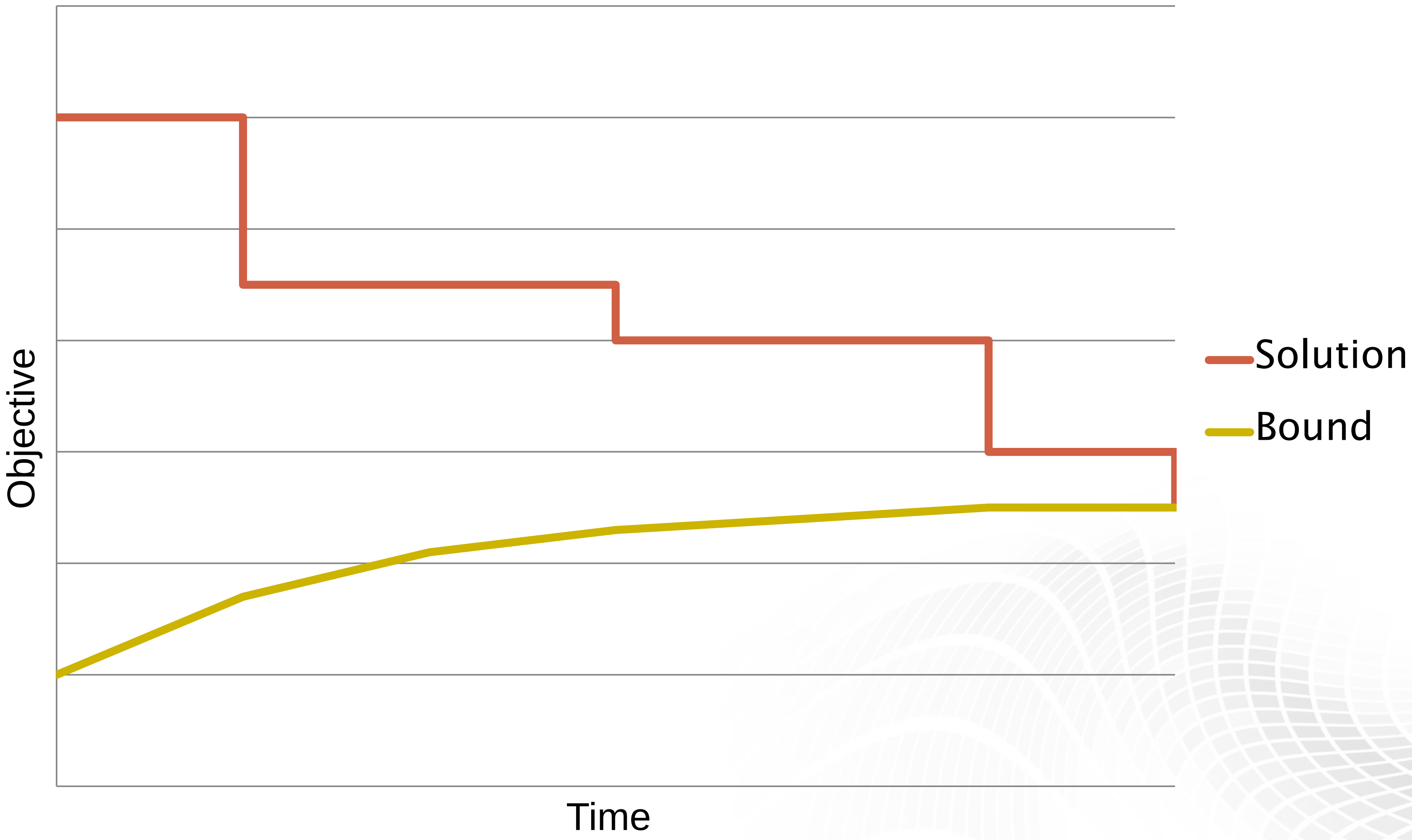


Remarks:

(1) GAP = 0 $\Rightarrow$ Proof of optimality

(2) In practice: good quality solution often enough

Solving a MIP Model



Branching Variable Selection

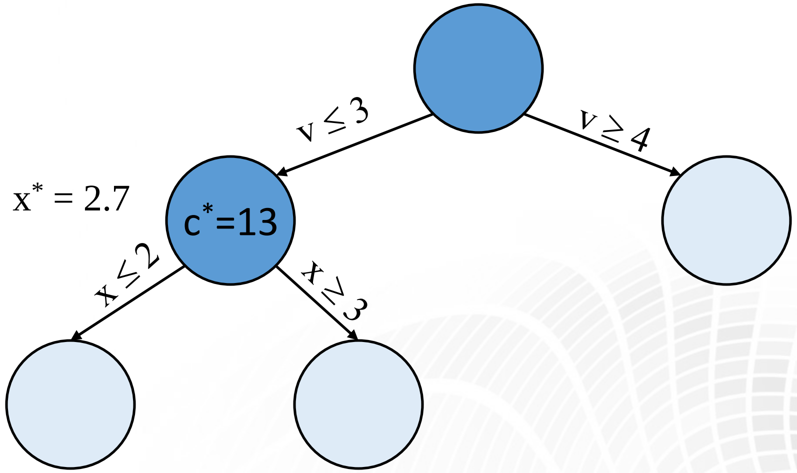
- Given a relaxation solution x^*
  - Branching candidates:
    - Integer variables x_j that take fractional values
      - $x_j = 3.7$ produces two child nodes ($x \leq 3$ or $x \geq 4$)
  - Need to pick a variable to branch on
    - Choice is crucial in determining the size of the overall search tree

Branching Variable Selection

- What's a good branching variable?
  - Superb: fractional variable infeasible in both branch directions
  - Great: infeasible in one direction
  - Good: both directions move the objective
- Expensive to predict which branches lead to infeasibility or big objective moves
  - Strong branching
    - Truncated LP solve for every possible branch at every node
    - Rarely cost effective
  - Need a quick estimate

Pseudo-Costs

- Use historical data to predict impact of a branch:
  - Record $\text{cost}(x_j) = \Delta\text{obj} / \Delta x_j$ for each branch
  - Store results in a pseudo-cost table
    - Two entries per integer variable
      - Average down cost
      - Average up cost
  - Use table to predict cost of a future branch



Pseudo-Costs

- Use historical data to predict impact of a branch:
  - Record $\text{cost}(x_j) = \Delta\text{obj} / \Delta x_j$ for each branch
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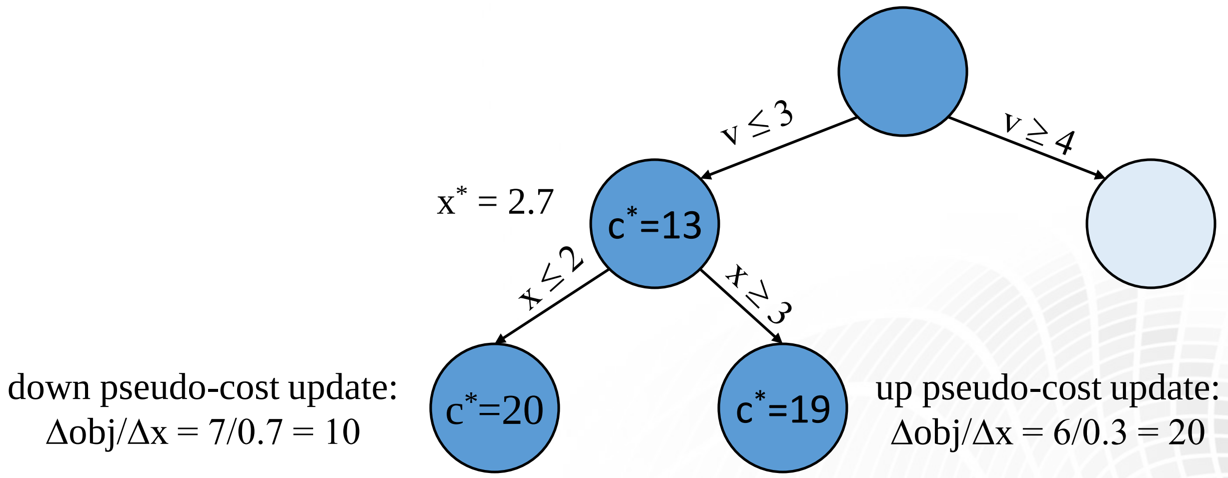

Pseudo-Costs

- Use historical data to predict impact of a branch:
  - Record $cost(x_j) = \Delta obj / \Delta x_j$ for each branch
  - Store results in a pseudo-cost table
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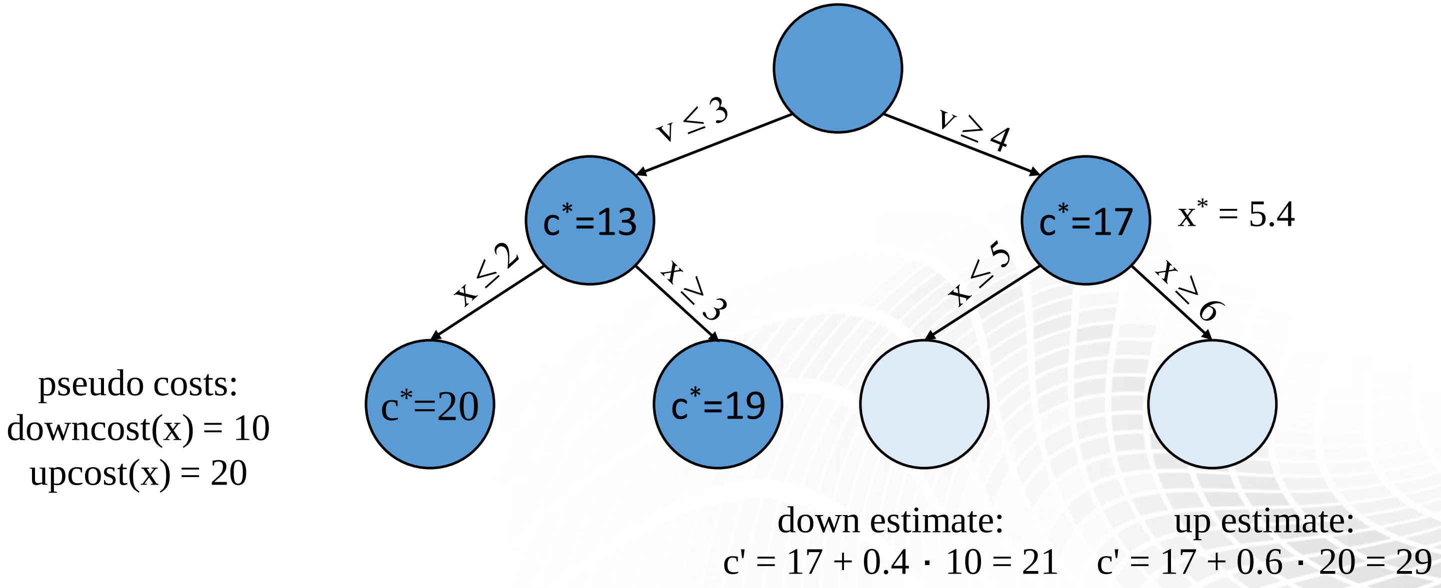
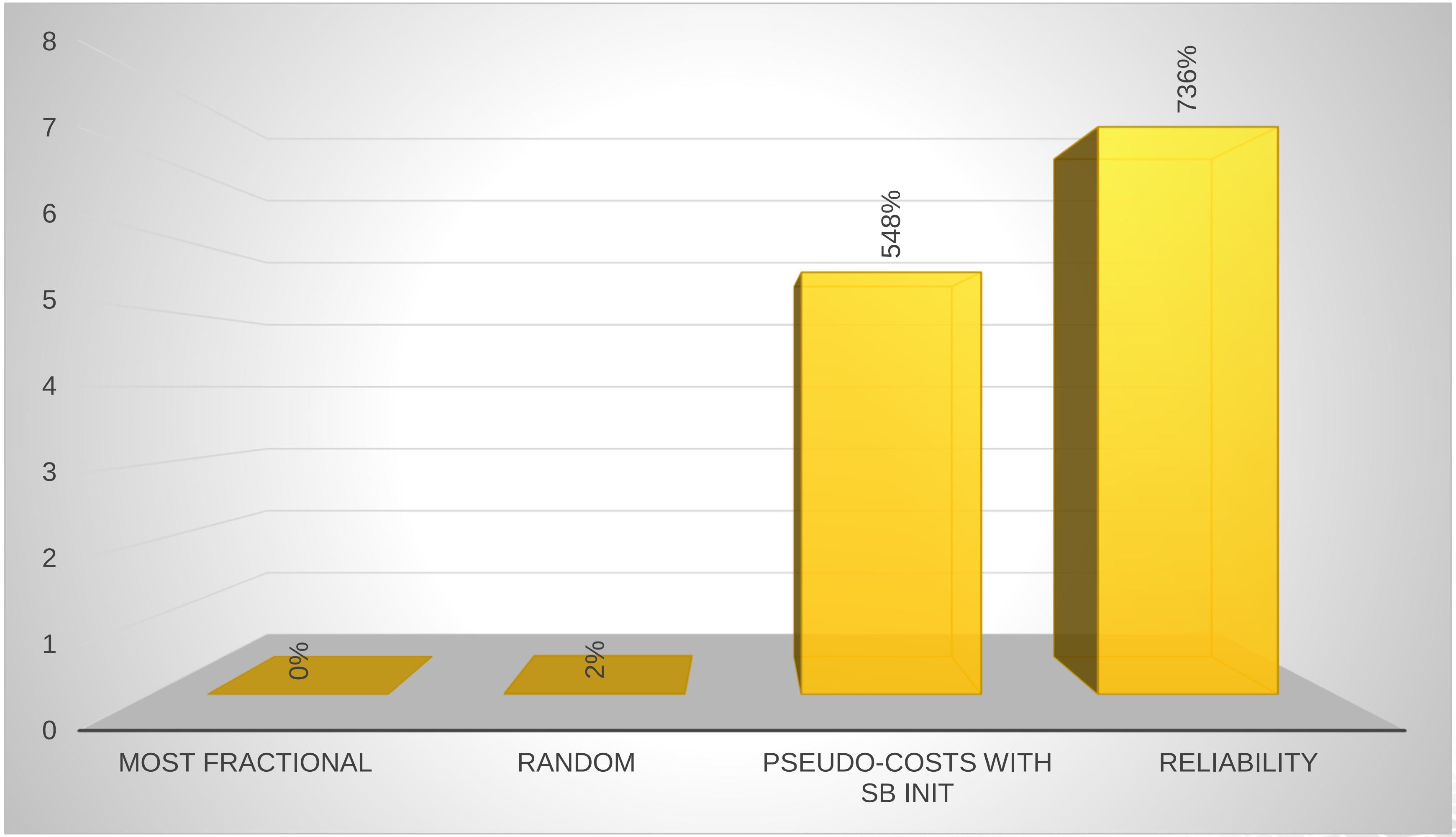
      - Average up cost
  - Use table to predict cost of a future branch



Pseudo-Costs Initialization

- What do you do when there is no history?
  - E.g., at the root node
- Initialize pseudo-costs [Linderoth & Savelsbergh, 1999]
  - Always compute up/down cost (using strong branching) for new fractional variables
    - Initialize pseudo-costs for every fractional variable at root
- Reliability branching [Achterberg, Koch & Martin, 2005]
  - Do not rely on historical data until pseudo-cost for a variable has been recomputed r times

Branching Rules – Performance



Achterberg and Wunderling: "Mixed Integer Programming: Analyzing 12 Years of Progress" (2013)
benchmark data based on CPLEX 12.5
Achterberg, Koch, and Martin: "Branching Rules Revisited" (2005)

MIP Building Blocks

- Presolve
  - Tighten formulation and reduce problem size
- Solve continuous relaxations
  - Ignoring integrality
  - Gives a bound on the optimal integral objective
- Cutting planes
  - Cut off relaxation solutions
- Branching variable selection
  - Crucial for limiting search tree size
- Primal heuristics
  - Find integer feasible solutions

Primal Heuristics

- Try to find good integer feasible solutions quickly
  - Better pruning during search due to better bound
  - Reach desired gap faster
  - Often important in practice: quality of solution after fixed amount of time
- Start heuristics
  - Try to find integer feasible solution, usually "close" to LP solution
- Improvement heuristics
  - Given integer feasible solution, try to find better solution

Primal Heuristics Explained on Twitter

GUROBI OPTIMIZATION

Matteo Fischetti @MFischetti · 1 Std.
.@AchterbergT hmmm, challenging suggestion: write a full scientific paper in a tweet! I would be tempted...

Matteo Fischetti @MFischetti · 1 Std.
.@AchterbergT e.g. #LocalBranching take a 0-1 MIP and a solution x*, bound Hamming distance from x* through a linear cut, and solve again.

1

Matteo Fischetti @MFischetti · 1 Std.
.@AchterbergT #RINS take a MIP, a feasible sol. x~ and the LP relaxation sol. x*, fix all components that agree, and solve as a MIP.

1

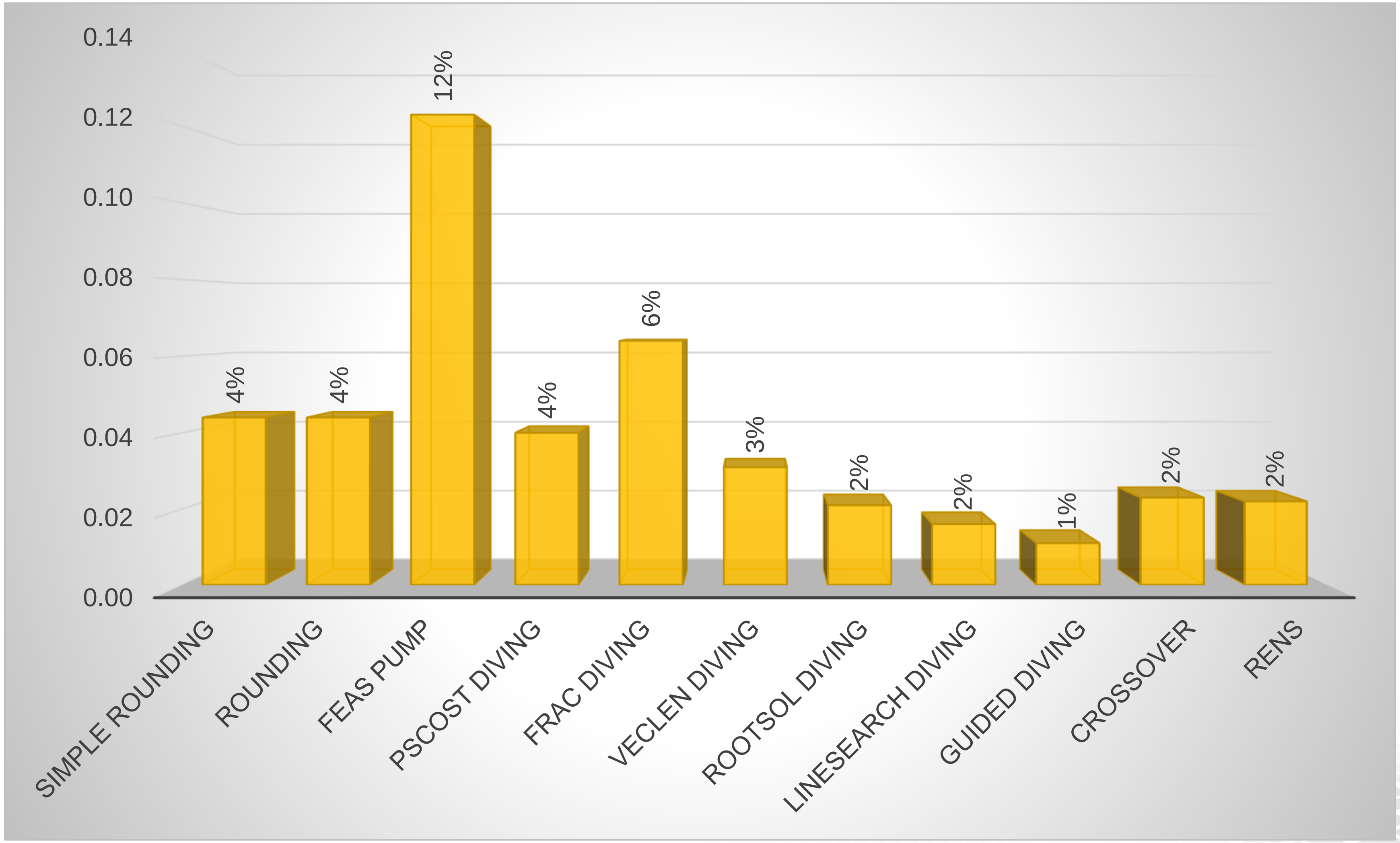
Matteo Fischetti @MFischetti · 1 Std.
.@AchterbergT #FP Take MIP and LP sol. x*, round it, solve LP again but minimizing Hamming distance from rounded sol. Repeat. Serve it warm.

2

Primal Heuristics – Examples

- Start heuristics
  - Rounding heuristics: round LP solution to integral values
    - Potentially, try to fix constraint infeasibilities
  - Fix-and-dive heuristics: fix variables, propagate, resolve LP
  - Feasibility pump: push LP solution towards integrality by modifying objective
  - RENS: Solve sub-MIP in neighborhood of LP solution
- Improvement heuristics
  - 1-Opt and 2-Opt: Modify one or two variables to get better objective
  - Local Branching: Solve sub-MIP in neighborhood of MIP solution
  - Mutation: Solve sub-MIP in neighborhood of MIP solution
  - Crossover: Solve sub-MIP in neighborhood of 2 or more MIP solutions
  - RINS: Solve sub-MIP in neighborhood of LP and MIP solution

Primal Heuristics – Performance

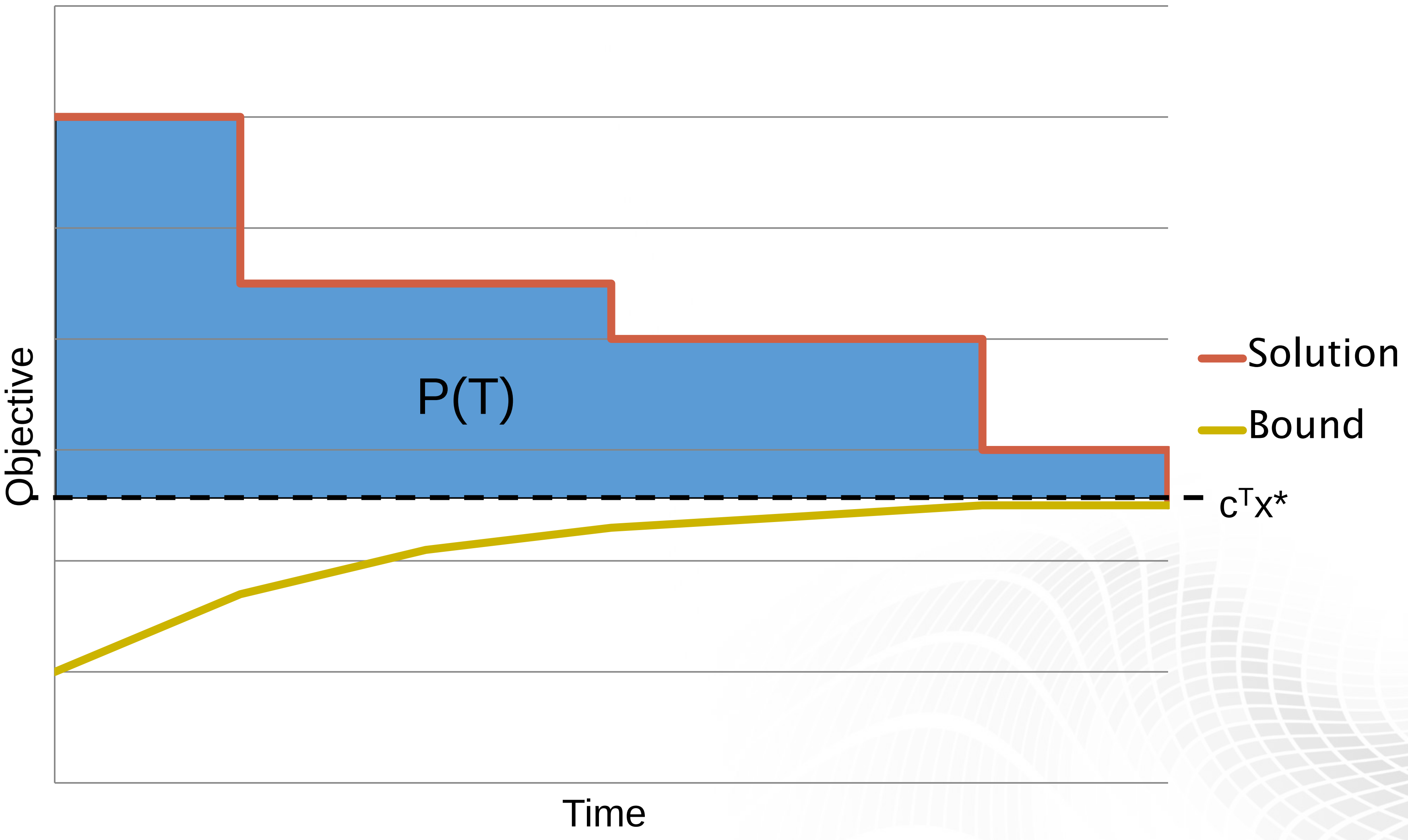


Berthold: "Primal Heuristics for Mixed Integer Programs" (2006)
benchmark data based on SCIP 0.82b

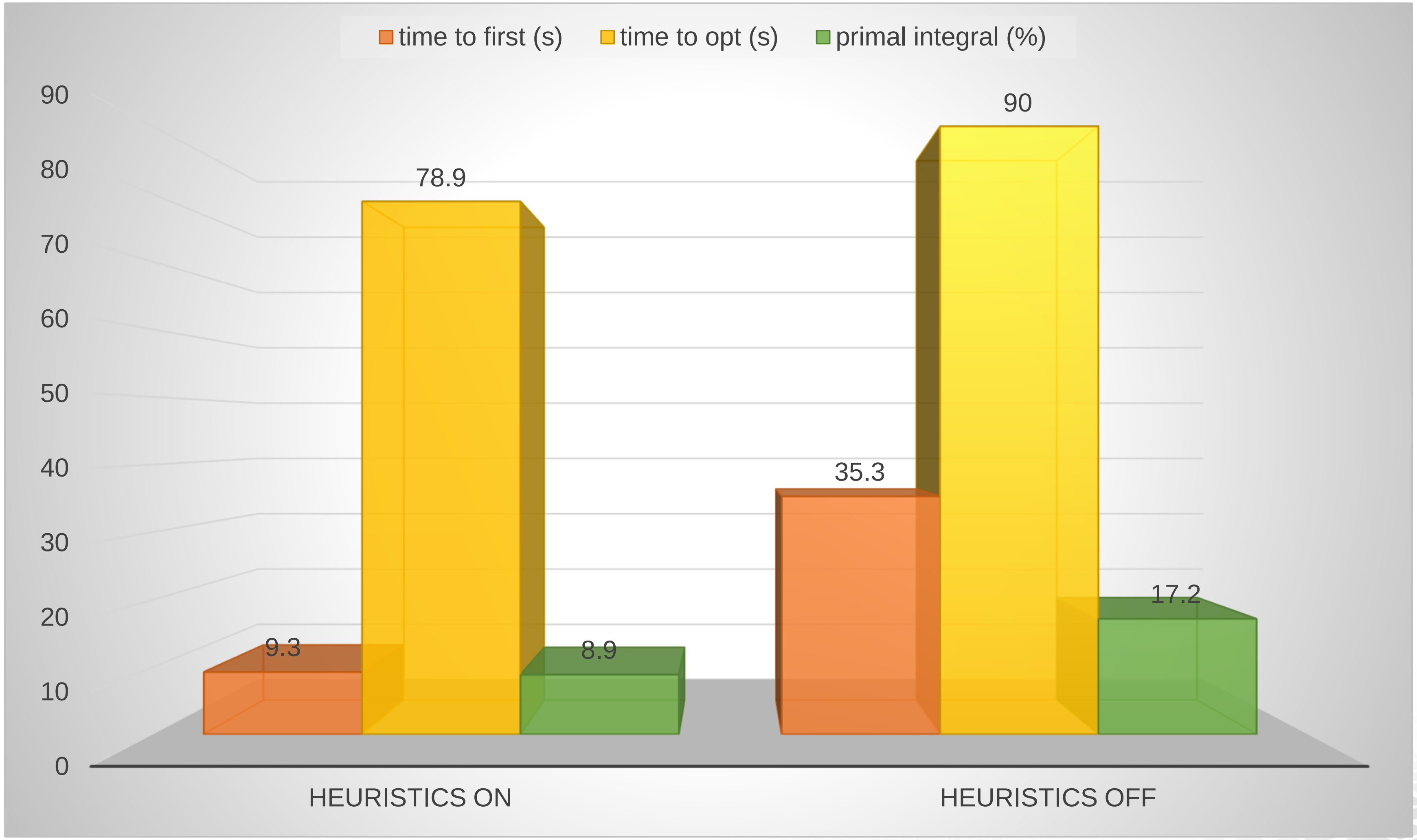
Primal Heuristics – Measuring Performance

- Is time to optimality a good measure to assess impact of heuristics?
  - Goal of heuristics is to provide good solutions quickly
  - Faster progress in dual bound due to additional pruning is only secondary
  - Often important for practitioners:
    - Find any feasible solution quickly to validate that model is reasonable
    - Find good solution in reasonable time frame
- Primal gap: $\gamma^p(\tilde{x}) = \frac{|c^T x^* - c^T \tilde{x}|}{max\{|c^T x^*|, |c^T \tilde{x}|\}}$
- Primal gap function: $p(t) = \begin{cases} 1, & \text{if no incumbent until time } t \\ \gamma^p(\tilde{x}(t)), & \text{with } \tilde{x}(t) \text{ being incumbent at time } t \end{cases}$
- Primal integral: $P(T) = \int_{t=0}^{T} p(t)dt$

Primal Integral



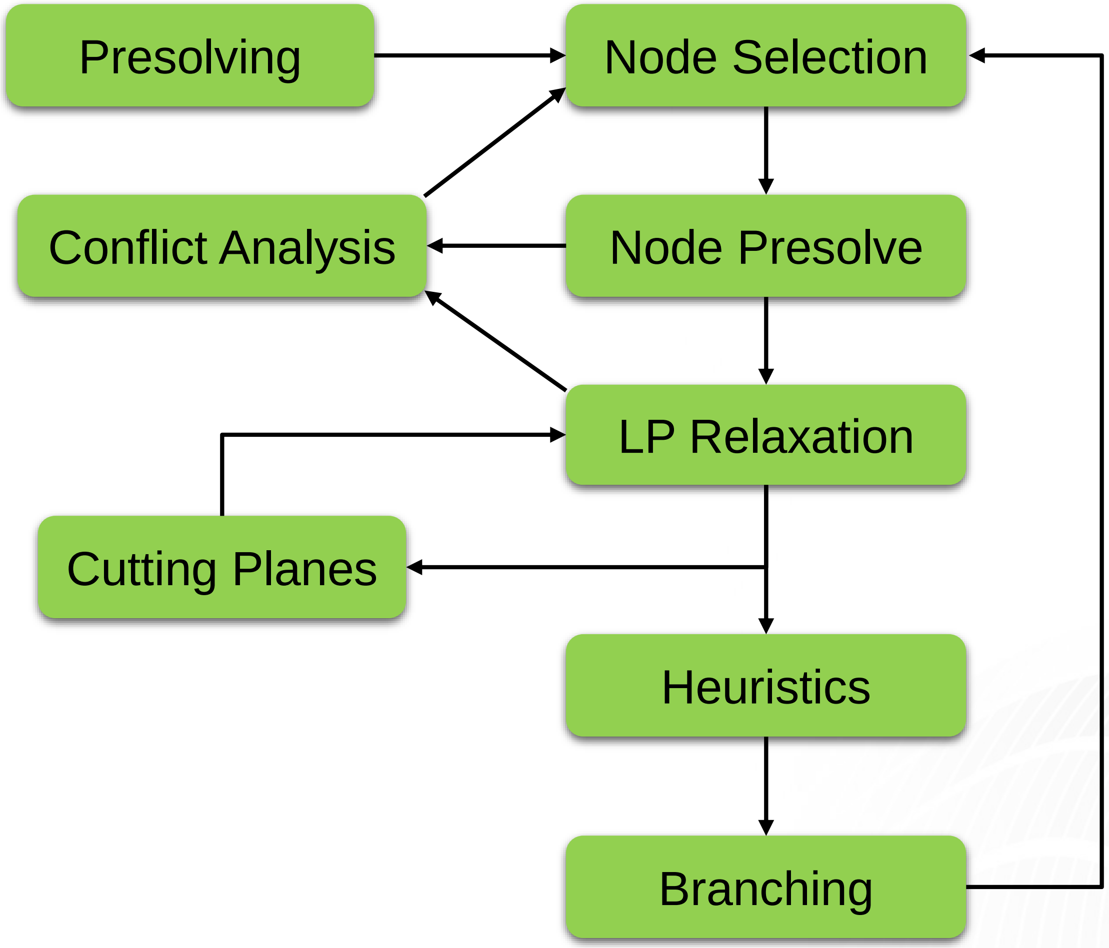
Primal Heuristics – Performance



Berthold (2014): "Heuristic algorithms in global MINLP solvers"
benchmark data based on SCIP 3.0.2

Putting It All Together

Branch-and-Cut



Branch-and-Cut

Presolving

Conflict Analysis

Cutting Planes

Branching

```
Gurobi Optimizer version 6.0.0 (linux64)
Copyright (c) 2014, Gurobi Optimization, Inc.

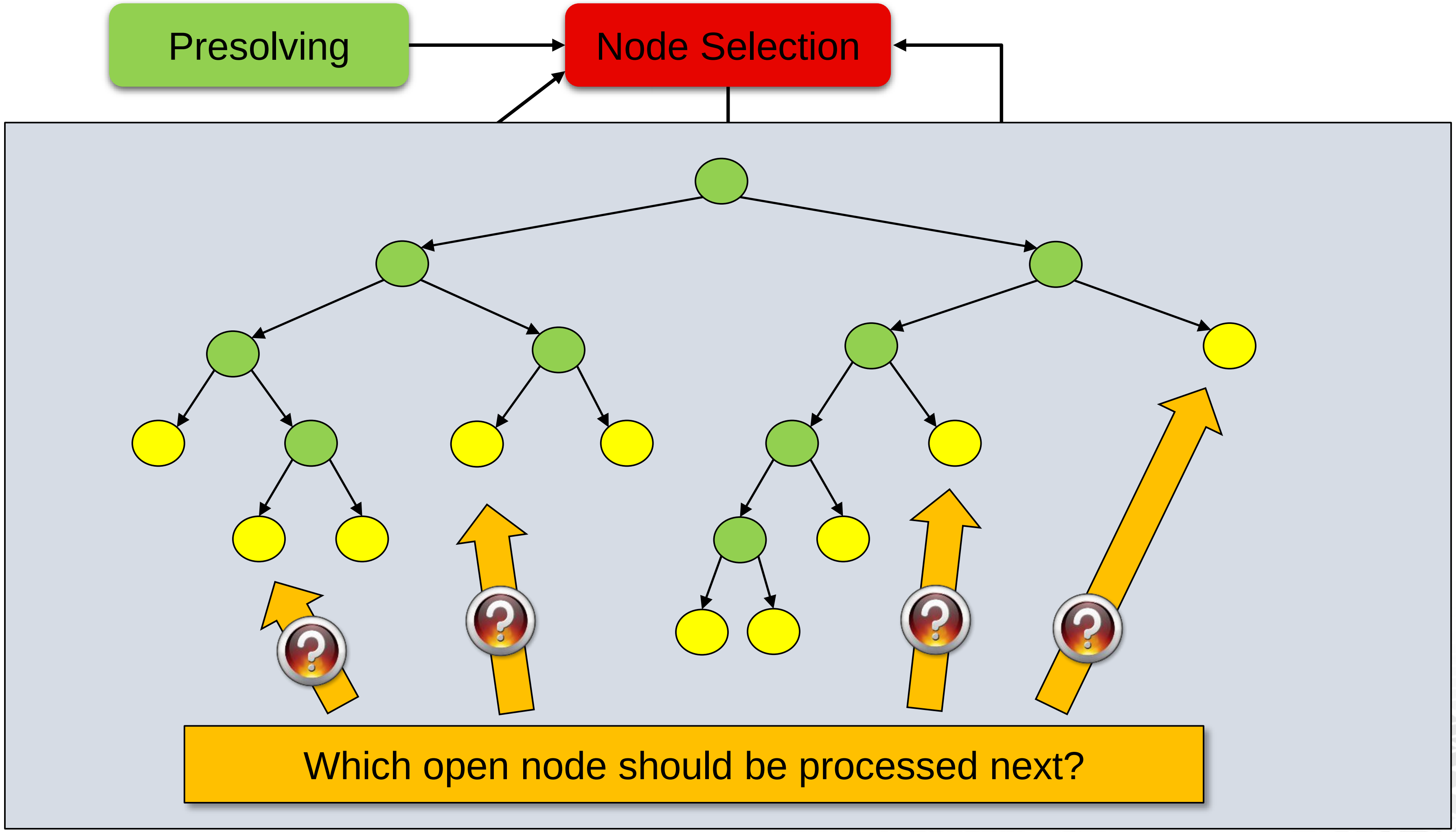
Read MPS format model from file /models/mip/roll3000.mps.bz2
Reading time = 0.03 seconds
roll3000: 2295 rows, 1166 columns, 29386 nonzeros
Optimize a model with 2295 rows, 1166 columns and 29386 nonzeros
Coefficient statistics:
  Matrix range    [2e-01, 3e+02]
  Objective range [1e+00, 1e+00]
  Bounds range    [1e+00, 1e+09]
  RHS range       [6e-01, 1e+03]
Presolve removed 1308 rows and 311 columns
Presolve time: 0.08s
Presolved: 987 rows, 855 columns, 19346 nonzeros
Variable types: 211 continuous, 644 integer (545 binary)

Root relaxation: objective 1.112003e+04, 1063 iterations, 0.03 seconds

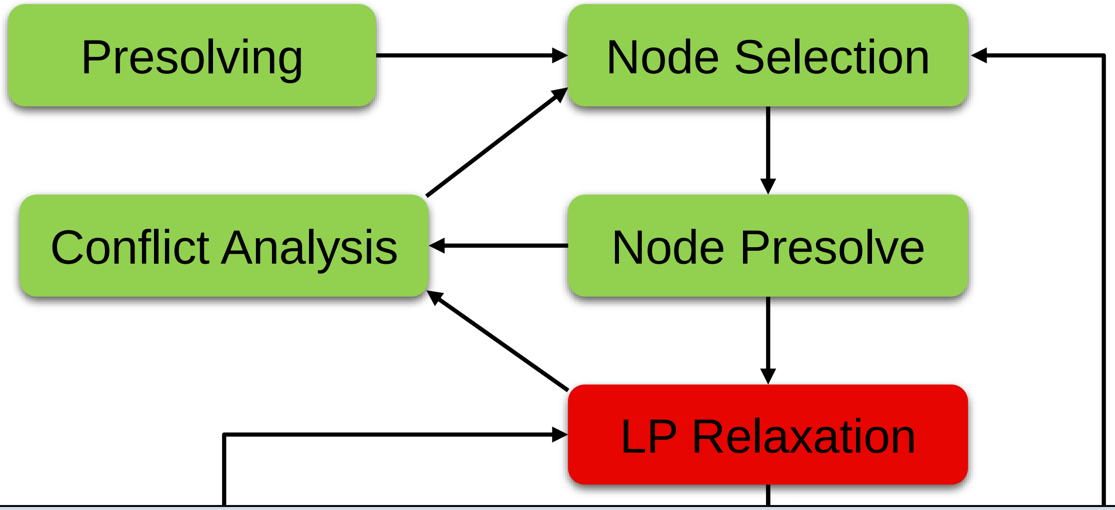
    Nodes    |    Current Node    |     Objective Bounds      |     Work
 Expl Unexpl |  Obj  Depth IntInf | Incumbent    BestBd   Gap | It/Node Time

     0     0 11120.0279    0  154          - 11120.0279     -      -    0s
     0     0 11526.8918    0  207          - 11526.8918     -      -    0s
     0     0 11896.9710    0  190          - 11896.9710     -      -    0s
```

Branch-and-Cut



Branch-and-Cut




```
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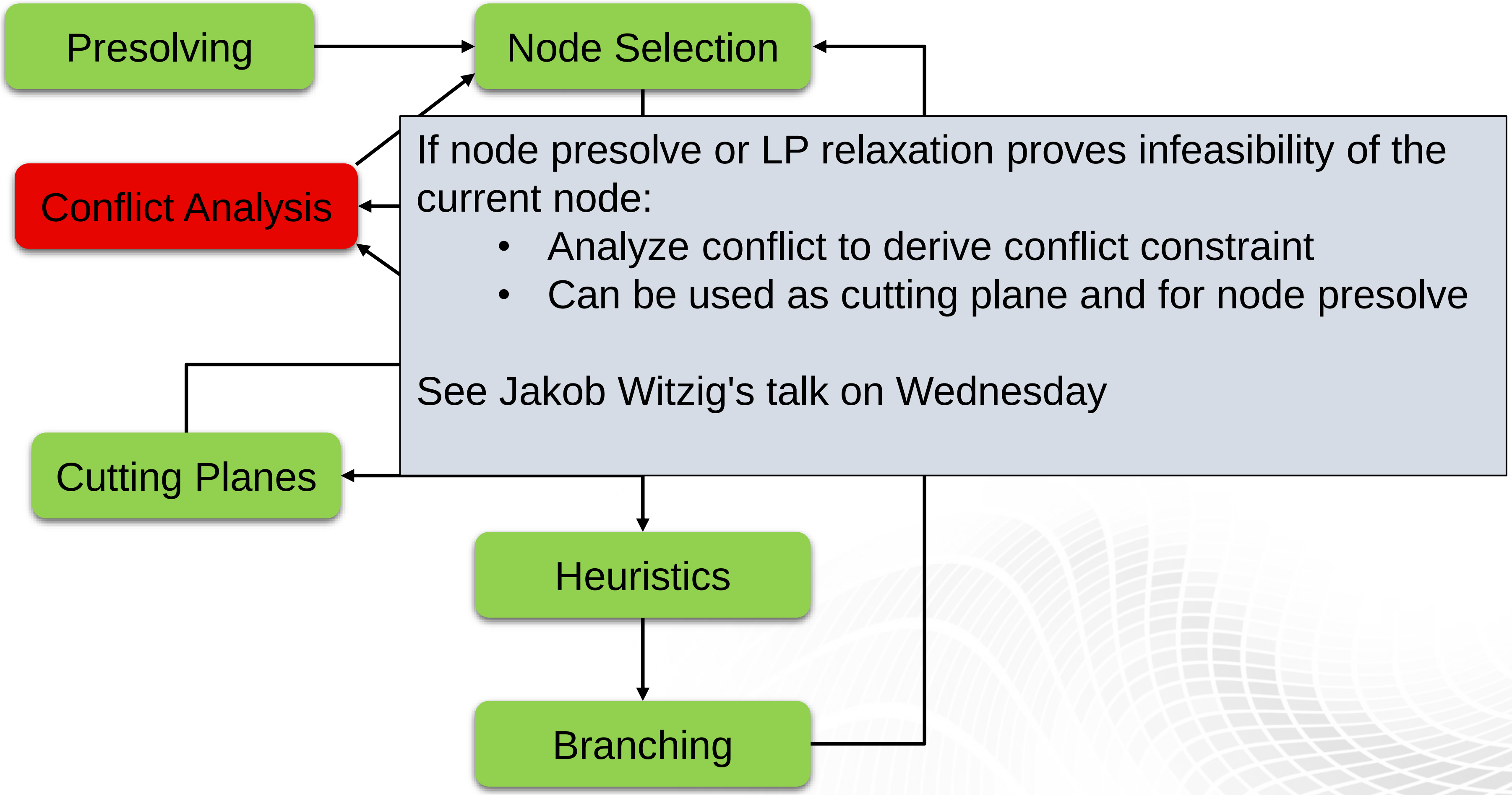
    Nodes    |    Current Node    |     Objective Bounds      |     Work
 Expl Unexpl |  Obj  Depth IntInf | Incumbent    BestBd   Gap | It/Node Time

     0     0 11120.0279    0  154          - 11120.0279     -      -    0s
     0     0 11526.8918    0  207          - 11526.8918     -      -    0s
     0     0 11896.9710    0  190          - 11896.9710     -      -    0s
...
H  327   218                    13135.000000 12455.2162  5.18%  42.6    1s
H  380   264                    13093.000000 12455.2162  4.87%  41.6    1s
H  413   286                    13087.000000 12455.2162  4.83%  41.4    1s
  1066   702 12956.2676   31  192 13087.0000 12629.5426  3.50%          5s
```

Branch-and-Cut

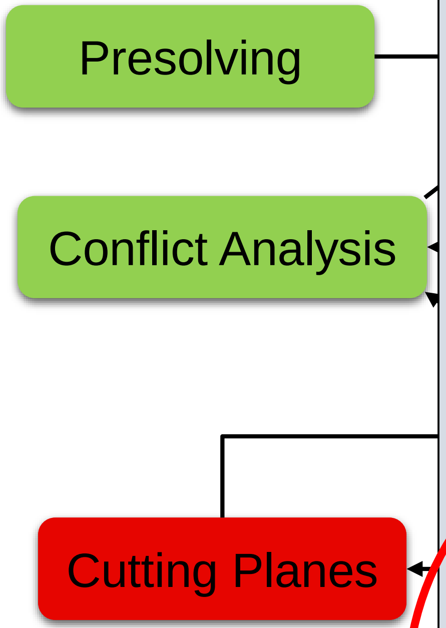


If node presolve or LP relaxation proves infeasibility of the current node:

- Analyze conflict to derive conflict constraint
- Can be used as cutting plane and for node presolve

See Jakob Witzig's talk on Wednesday

Branch-and-Cut

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 Expl Unexpl |  Obj  Depth IntInf | Incumbent    BestBd   Gap | It/Node Time

     0     0 11120.0279    0  154          - 11120.0279     -      -    0s
     0     0 11526.8918    0  207          - 11526.8918     -      -    0s
     0     0 11896.9710    0  190          - 11896.9710     -      -    0s
     0     0 12151.4022    0  190          - 12151.4022     -      -    0s
     0     0 12278.3391    0  208          - 12278.3391     -      -    0s
...
  5485   634 12885.3652   52  143 12890.0000 12829.0134  0.47%  54.5   25s

Cutting planes:
  Learned: 4
  Gomory: 46
  Cover: 39
  Implied bound: 8
  Clique: 2
  MIR: 112
  Flow cover: 27
  GUB cover: 11
  Zero half: 91

Explored 6008 nodes (357915 simplex iterations) in 27.17 seconds
Thread count was 4 (of 8 available processors)
```

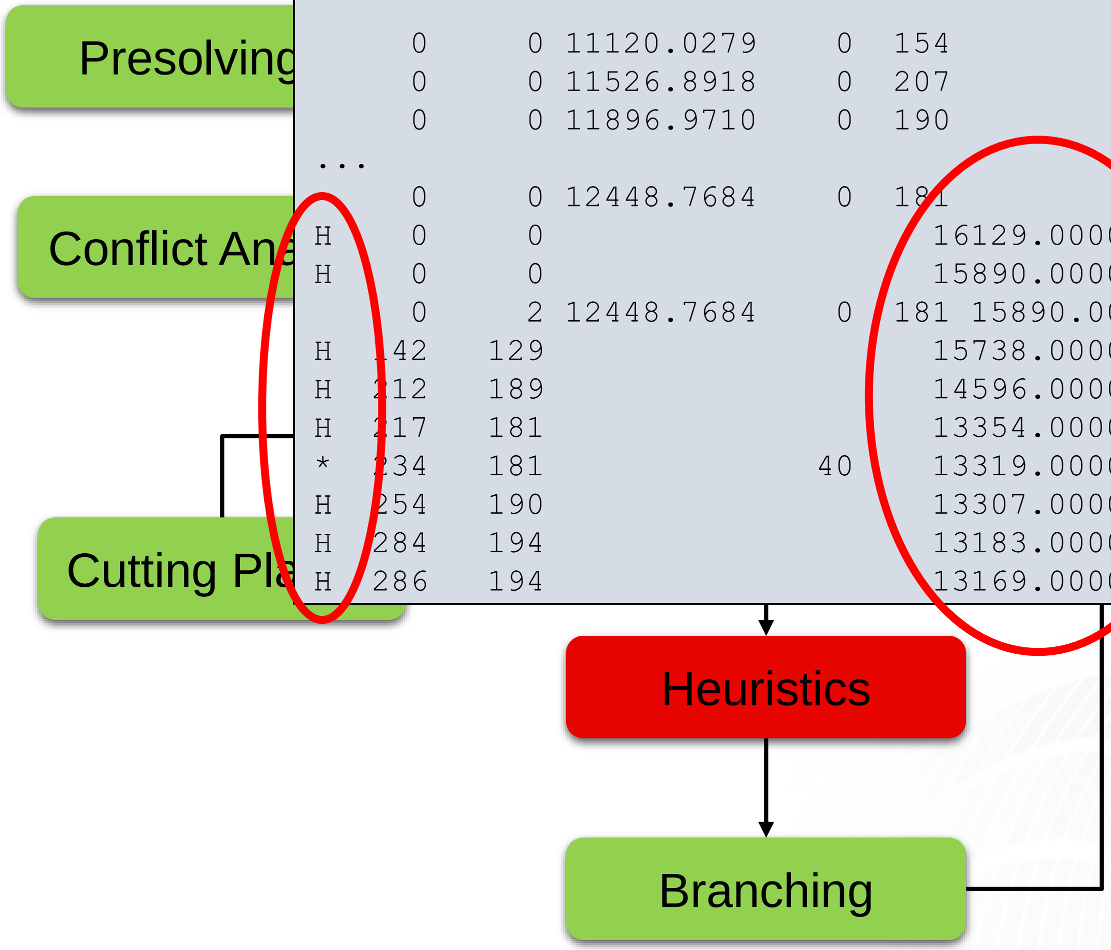
Branch-and-Cut

```
Presolved: 987 rows, 855 columns, 19346 nonzeros
Variable types: 211 continuous, 644 integer (545 binary)

Root relaxation: objective 1.112003e+04, 1063 iterations, 0.03 seconds

    Nodes    |    Current Node    |     Objective Bounds      |     Work
 Expl Unexpl |  Obj  Depth IntInf | Incumbent    BestBd   Gap | It/Node Time

     0     0 11120.0279    0  154          - 11120.0279      -     -    0s
     0     0 11526.8918    0  207          - 11526.8918      -     -    0s
     0     0 11896.9710    0  190          - 11896.9710      -     -    0s
...
     0     0 12448.7684    0  181          - 12448.7684      -     -    0s
H    0     0                    16129.000000 12448.7684  22.8%     -    0s
H    0     0                    15890.000000 12448.7684  21.7%     -    0s
     0     2 12448.7684    0  181 15890.0000 12448.7684  21.7%     -    0s
H  142   129                    15738.000000 12450.7195  20.9%  43.8    1s
H  212   189                    14596.000000 12453.8870  14.7%  42.3    1s
H  217   181                    13354.000000 12453.8870  6.74%  42.6    1s
*  234   181             40    13319.000000 12453.8870  6.50%  42.1    1s
H  254   190                    13307.000000 12453.8870  6.41%  41.3    1s
H  284   194                    13183.000000 12453.8870  5.53%  42.6    1s
H  286   194                    13169.000000 12453.8870  5.43%  42.7    1s
```



Branch-and-Cut

Presolvi

Conflict An

Cutting P

```
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Root relaxation: objective 1.112003e+04, 1063 iterations, 0.03 seconds

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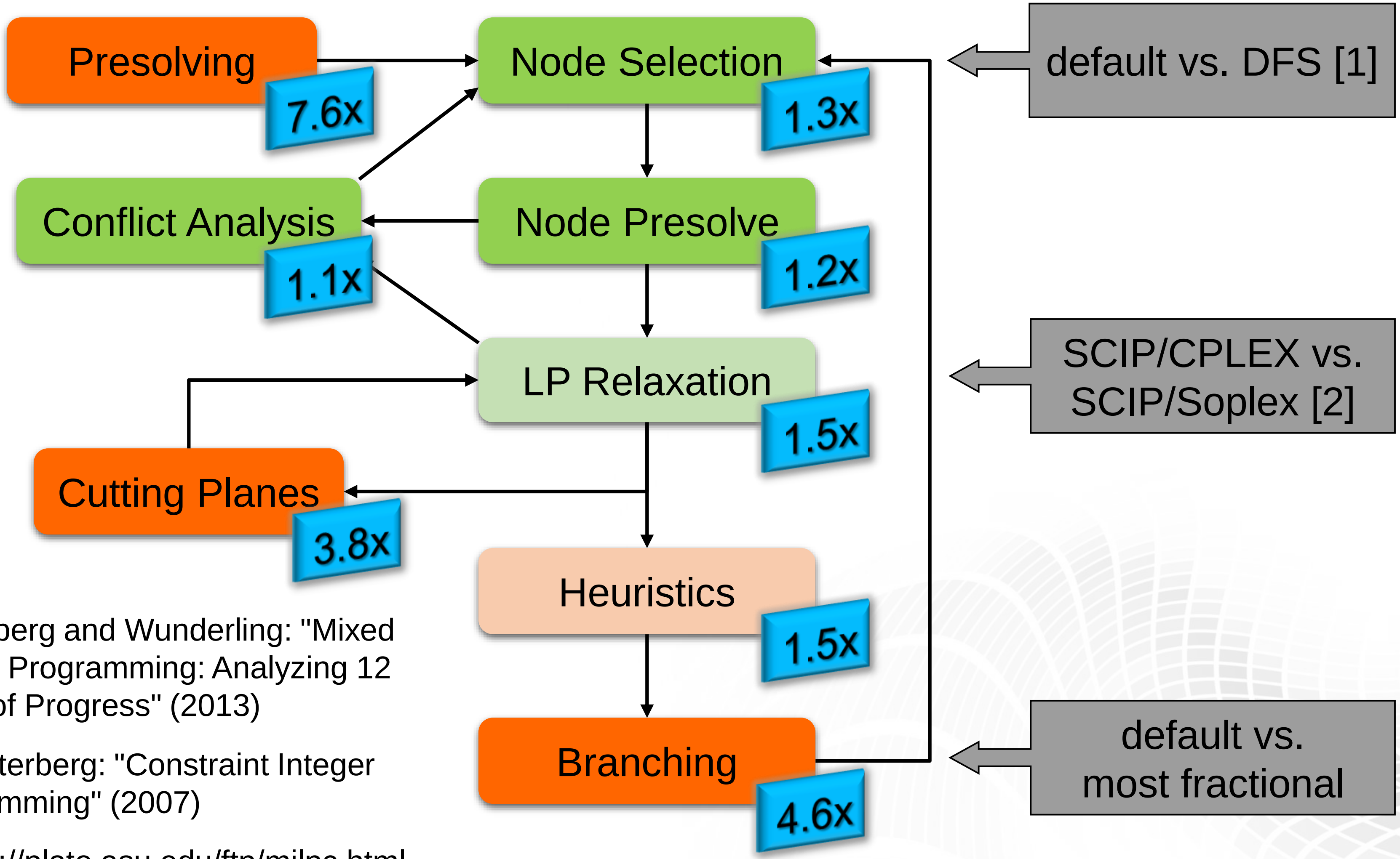
 Expl Unexpl |  Obj  Depth IntInf | Incumbent    BestBd   Gap | It/Node Time

     0     0 11120.0279    0  154          - 11120.0279     -      -    0s
     0     0 11526.8918    0  207          - 11526.8918     -      -    0s
     0     0 11896.9710    0  190          - 11896.9710     -      -    0s
...
H    0     0                    15890.000000 12448.7684  21.7%     -    0s
     0     2 12448.7684    0  181 15890.0000 12448.7684  21.7%     -    0s
...
  1066   702 12956.2676   31  192 13087.0000 12629.5426  3.50%  37.2    5s
  1097   724 12671.8285    8  147 13087.0000 12671.8285  3.17%  41.6   10s
  1135   710 12732.5601   32  126 12890.0000 12727.1362  1.26%  44.6   15s
  3416   887 12839.9880   46  136 12890.0000 12780.7059  0.85%  49.7   20s
  5485   634 12885.3652   52  143 12890.0000 12829.0134  0.47%  54.5   25s
```

Branching

Performance Impact of MIP Solver Components (CPLEX 12.5 or SCIP)



Achterberg and Wunderling: "Mixed Integer Programming: Analyzing 12 Years of Progress" (2013)

[1] Achterberg: "Constraint Integer Programming" (2007)

[2] http://plato.asu.edu/ftp/milpc.html

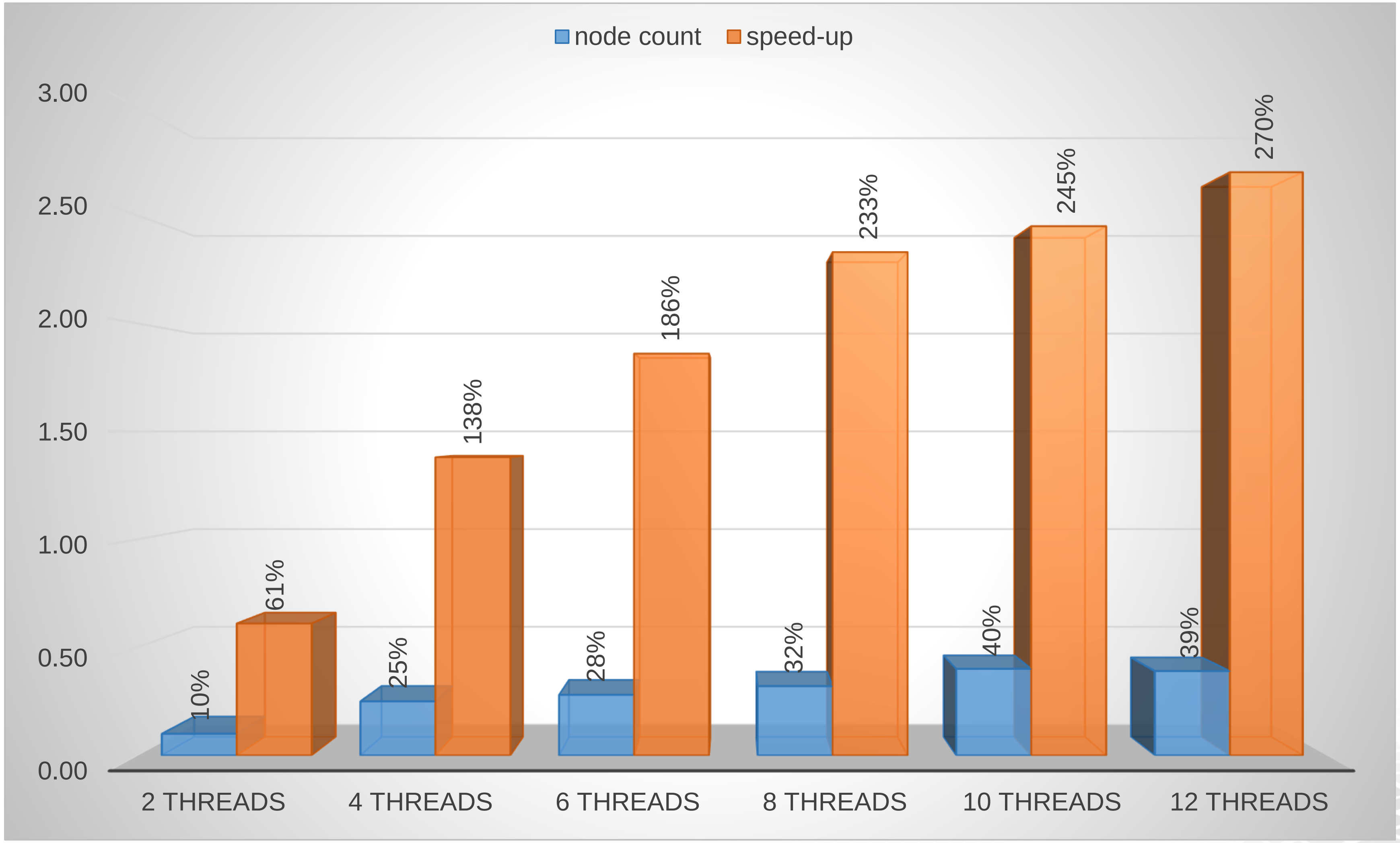
Parallelization

- Parallelization opportunities
  - Parallel probing during presolve
    - Almost no improvement
  - Use barrier or concurrent LP for initial LP relaxation solve
    - Only helps for large models
  - Run heuristics or other potentially useful algorithms in parallel to the root cutting plane loop
    - Moderate performance improvements: 20-25%
    - Does not scale beyond a few threads
  - Solve branch-and-bound nodes in parallel
    - Main speed-up for parallel MIP
    - Performance improvement depends a lot on shape of search tree
    - Typically scales relatively well up to 8 to 16 threads

Parallelization

- Parallelization issues
  - Determinism
  - Load balancing
  - CPU heat and memory bandwidth
    - Additional threads slow down main thread
  - Root node does not parallelize well
    - Sequential runtime of root node imposes limits on parallelization speed-up
    - Amdahl's law
  - A dive in the search tree cannot be parallelized
    - Parallelization only helps if significant number of dives necessary to solve model

Parallel MIP – Performance



Achterberg and Wunderling: "Mixed Integer Programming: Analyzing 12 Years of Progress" (2013)
benchmark data based on CPLEX 12.5, models with ≥ 100 seconds solve time

No Further Questions? Enjoy Your Coffee Break!